AD-A044 774 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 13/11

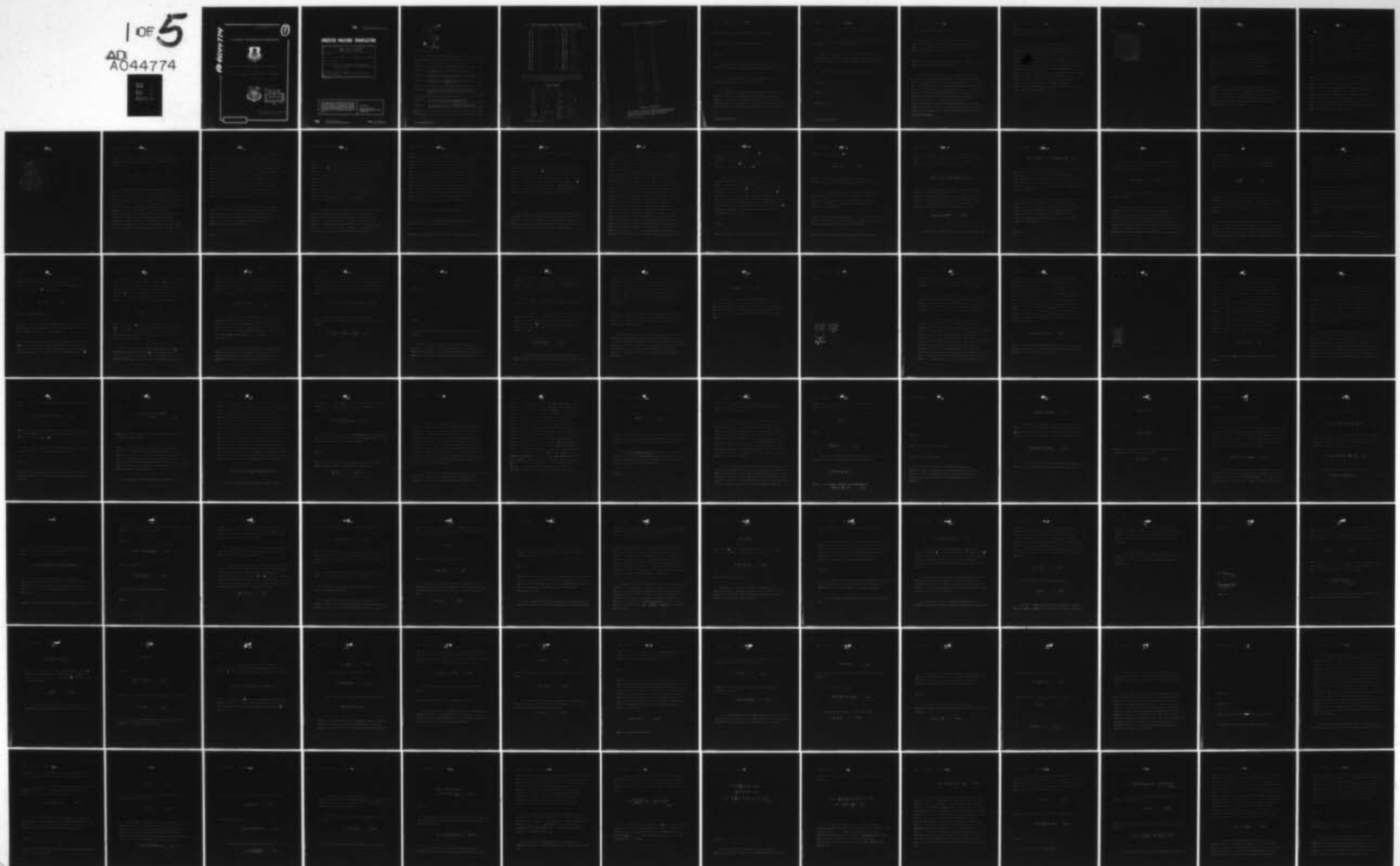
THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS (TEORIY--ETC(U)

JAN 77 G I BARENBLATT, V M YENTOV, V M RYZHIK

UNCLASSIFIED FTD-ID(RS)T-1860-76-PT-1 NL

1 OF 5

AD A044774



AD-A044774

FTD-ID(RS)T-1860-76
PART 1 OF 2

①

FOREIGN TECHNOLOGY DIVISION

THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS

by

G. I. Barenblatt, V. M. Yentov, V. M. Ryzhik

FTD- ID(RS)T-1860-76

UNEDITED MACHINE TRANSLATION

FTD-ID(RS)T-1860-76 24 January 1977

FTD-77-C-000070

THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS

By: G. I. Barenblatt, V. M. Yentov, V. M. Ryzhik

English pages: 931

Source: Teoriya Nestatsionarnoy Fil'tratsii Zhidkosti i Gaza, Izd-vo "Nedra," Moscow, 1972, PP. 1-288.

Country of origin: USSR
This document is a machine aided translation.
Requester: NSWC

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

Table of Contents

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

GREEK ALPHABET

Alpha	Α	α	α		Nu	Ν	ν	
Beta	Β	β			Xi	Ξ	ξ	
Gamma	Γ	γ			Omicron	Ο	ο	
Delta	Δ	δ			Pi	Π	π	
Epsilon	Ε	ε	ϵ		Rho	Ρ	ρ	ϱ
Zeta	Ζ	ζ			Sigma	Σ	σ	ς
Eta	Η	η			Tau	Τ	τ	
Theta	Θ	θ	ϑ		Upsilon	Υ	υ	
Iota	Ι	ι			Phi	Φ	φ	ϕ
Kappa	Κ	ϰ	κ	ϰ	Chi	Χ	χ	
Lambda	Λ	λ			Psi	Ψ	ψ	
Mu	Μ	μ			Omega	Ω	ω	

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

GRAPHICS DISCLAIMER

All figures, graphics, tables, equations, etc. merged into this translation were extracted from the best quality copy available.

THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS.

G. I. Barenblatt, V. M. Yentov, V. M. Ryzhik.

Page 2.

Barenblatt G. I., Yentov V. M., Ryzhik V. m. Theory of the unsteady filtration of liquid and gas. M., "interiors", 1972, s. 288.

In the book is given the short characteristic of the porous media in which occurs the filtration of liquid.

Basic attention is devoted to the solution of problems in the unsteady filtration of liquid, gas and multicomponent systems. Are examined the theoretical prerequisite/premises of filtration in the porous, cracked and cracked-porous media. Are described the laws of the filtration of the mixtures of different physical properties,

dependence of the displacement of some liquids by others.

All problems are solved in connection with the development of petroleum, gas and gas-condensation deposits.

The book is intended for the technical personnel of the oil-extracting industry. Special interest it will be of for the scientific workers of scientific-research and designed institutes.

Tables 7, illustrations 89, bibliography is 162 names.

Page 3.

Chapter I.

PHYSICAL BASES.

1. Porous medium and its properties.

The filtration theory studies the motion of gases, liquids and their mixtures in the porous media, i.e., in solids, pierced by the system of the communicating voids (pores), which makes them those which are penetrated for liquids [1].

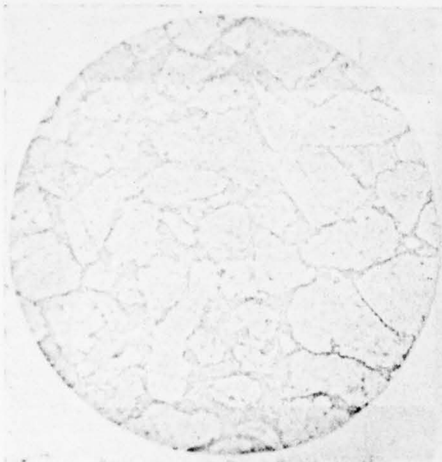
FOOTNOTE [1]. It goes without saying that part of the pores can be that which was isolate/insulated. ENDFOOTNOTE.

The motion of liquids and gases in the porous medium has a series of special feature/peculiarities. The porous medium consists of the enormous number of randomly arranged/located grains of various forms and value. Therefore the space in which moves the liquid, is the system of the pores, which continuously convert one to another. For the porous medium characteristically the property of the communication of pores, it it is not possible to visualize in the form of the totality of the capillaries, arrange/located independently one from another. Certain concept about the porous medium gives the photograph of the section of oil-bearing sandstone (Fig. I.1). The character of communication/connection of the pores

between with is observable in the photograph of the mold of pore space (Fig. I.2), borrowed from book [58].

The irregular character of the structure of pore space does not make it possible to study the motion of liquid and gases in it by the direct/straight application/use of usual methods of hydrodynamics, i.e., means to the solution of the equations of motion of viscous fluid for region, which is the totality of all pores. This soluti ctly, even the notation of the boundary conditions of this problem), obviously, connected with insurmountable difficulties. However, in this solution there is no need: with an increase in the number of separate micromotions, constituting macroscopic filtration motion, begin to be exhibited the total statistical laws, characteristic for motion as a whole and not valid for one pore channel or several channels. The appearing situation is characteristic for systems with the large number of cell/elements (see 1 books [66]), weakly connected.

Fig. I.1.



Such systems can be described as some continuous media whose properties are not expressed directly through the properties of constituent elements, but are the averaged characteristics sufficient large volumes the media.

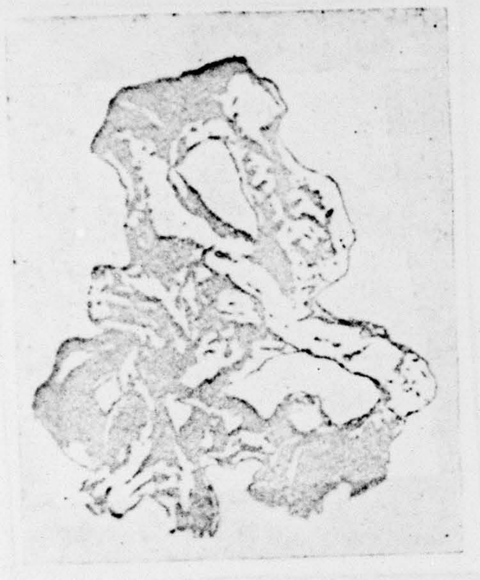
Similar to this in hydrodynamics is not examined the motion of separate molecules, but they are introduced some dynamic characteristics of liquid as continuous medium. With this approach by hydrodynamics are examined the only volumes of liquids whose size/dimensions are sufficiently great in comparison with intermolecular distances in order that in any volume would be located the sufficiently large number of molecules and would be possibly averaging [1].

FOOTNOTE [1]. As is known from the probability theory, the greater the number random sublimity, the generatrices certain/some totality, the lesser the probability of the deviation of the average value of the parameter for the given realization from most probable value. Thereby the indicated requirements make integral motion characteristics sufficiently stable. ENDFOOTNOTE.

Analogously filtration theory is constructed on the concept about about the fact that the porous medium and its filling liquid form continuous medium. This means that the physically infinitesimal system elements the liquid-porous medium all the same are sufficiently great in comparison with the size/dimensions of pores and grains of the porous medium; only for the volume in which included large number of pores and grains, are sufficiently representative the introduced averaged characteristics. In application to less volumes, the conclusion/derivations of the theory of filtration lose force.

From the viewpoint of filtration theory the value of the solid skeleton of the porous medium first of all geometric: it limits that region of space, in which moves the liquid. Only in the more special cases about which it will be said below, it is necessary to directly consider power interaction between the skeleton and the adjacent to it layers of liquid. Therefore the properties of the porous medium in filtration theory are described by certain set of dimensional characteristics. Due to the irregularity of the structure of its pore space it is not possible to completely describe by any final set of parameters; for purposes of filtration theory, however, sufficient small number of averaged characteristics.

Fig. I.2.



The most important characteristic of the porous medium - its porosity m, is equal to the ratio of the volume, occupied in the chosen cell/element with burrows, to the total volume of the cell/element:

$$m = V_n/V. \qquad (1.1.1)$$

The relationship/ratio (1.1.1) determines the average porosity of this cell/element. After selecting certain point of the porous medium, surrounding it by cell/elements increasingly less volume, we can define local porosity as limit of porosity with the contraction of volume. In this case, it is necessary to keep in mind that with the "contraction" of cell/element it always must remain large in comparison with the microscale of the porous medium (by size/dimension of pores or grains). Situation here is completely analogous to position in other sections of the mechanics of continuous medium; so, during the determination of the local density of gas the size/dimension of control volume always is selected large in comparison with intermolecular distances [see, for exam., 95].

During the determination of porosity, usually is distinguished the complete porosity in which are considered all pores, and the active porosity, during determination of which are considered only the those pores, which enter in the single system of interconnected pores and, therefore, they can be filled by liquid from without. For our purposes is important, it is natural, only active porosity; therefore subsequently by porosity, is understood namely it. Along with porosity m, sometimes is introduced the concept of the "prosvetnosti" of n, defined for each section, passing through this point as relation of the area of pores in section to an entire sectional area. It is easy to ascertain that the translucence at the particular point does not depend on the selection of the direction of section and is equal to porosity m [94].

The porosity characterizes form and the relative location of pores and is identical for the geometrically similar media. Along with porosity for the description of the porous medium, it is necessary to indicate also certain significant dimension of the pore space d_0. There are many actually equivalent methods of the determination of this size/dimension.

Page 6.

It is logical, for example, as the significant dimension d_0 to accept certain medium size of the pore channel d or of separate grain of porous skeleton 2. In order to calculate these medium size, in each concrete/specific/actual case is investigated the microstructure of the porous medium in certain sufficiently representative cell/element (volume or sections). By at first one method or the other is determined the size/dimension of separate pore or separate grain. This size/dimension is changed upon transition from one pore to another or from one grain to another. Therefore the results of measurements are represented in the form the distribution curve of the selected random size/dimension; the average value is obtained as result of certain averaging of distribution curve.

By themselves the distribution curves of the size/dimensions of pores or grains give considerably more information about the microstructure of the porous medium, than some average values. Therefore are undertaken numerous attempts at determining all geometric and hydrodynamic characteristics of the porous medium on the basis of distribution curves. However, the dependences of the characteristics of the porous medium on the parameters of

distribution curve cannot be universal. Actually, introducing, for example, fine/thin impenetrable partition/baffles, it is possible to radically change the hydrodynamic characteristics of the medium, having weakly changed the form of distribution curve. At the same time it is possible to indicate a series of processes (first of all the processes of the transfer in the porous medium) for which is essential the degree of the heterogeneity of the components of the porous medium - pores and grains. In this case, in addition to the average value of size/dimension is essential its dispersion, which characterizes the degree of deviation from the average value; usually it is assumed that distribution curve takes certain standard form (for example it is logarithmic normal) and it it is possible to completely describe, after assigning two parameter. Detailed information on this question can be found in book [3].

2. Law of the filtration of homogeneous liquid.

Filtration is the motion of liquid in the porous medium under the action of pressure differential [1].

FOOTNOTE [1]. As all values, the pressure of liquid it is assumed to be

averaged over the elementary macrovolume, which surrounds this point of the porous medium. ENDFOOTNOTE.

The fundamental characteristic of filtration motion is the velocity vector of the filtration of $\vec{u}$ determined as follows. Let us select point M of the porous medium and let us conduct through it surface element ΔS. Through the chosen area/site per unit time, flow/lasts the mass of liquid ΔQ. Then the projection of the vector of $\vec{u}$ on standard to the chosen area/site is equal $\lim_{\Delta S \to 0} \Delta Q/(\rho \Delta S)$, where ρ - the density of liquid. Let us stress that the mass of liquid is divided into complete area ΔS, but not into its part, occupied with pores.

Page 7.

The fundamental principles of the theory of filtration (law of filtration) establish/installs communication/connection between the velocity vector of filtration and that pressure field which causes filtration motion. Some information about the law of filtration can be obtained, on the strength of the quite general ideas.

Let us surround the point of the porous medium by certain small vicinity; the field of the rates of filtration in this vicinity it is possible to consider continuous, and all parameters of the porous medium and its saturating liquid - by constants. It is not possible to disregard only a change in the pressure, as little it not it was, since at constant along space pressure the motion completely is absent (actually this affirmation is the basic hypothesis). Since the pressure change in the vicinity of this point is determined by pressure gradient, the basic assumption during the establishment of the form of the law of filtration lies in the fact that the velocity vector of filtration at the particular point of the porous medium is determined by the properties of the liquid and porous medium and by pressure gradient grad p. Porous medium is characterized by geometric parameters - by the significant dimension d and by some dimensionless characteristics: porosity m, by the dimensionless parameters of the curve of distribution, etc. The law of filtration must be the consequence of the equations of the momentum of liquid in pore space; therefore into the system of the determining values, one should include/connect also those characteristics of liquids, which enter in these equations, i.e., density ρ and ductility/toughness/viscosity μ. Thus, it is assumed that there is a dependence of pressure gradient grad p on the velocity vector of the filtration of the aaaa of the dimensional characteristics of the porous medium m, d etc. and the characteristics of liquid ρ and μ. Among the values on which depends

grad p, the only rate of filtration of $\vec{u}$ is vector. On the strength of the isotropy of the medium (i.e. the independence of its properties from the rotation of the reflections of reference system) dependence grad p on $\vec{u}$, must be invariant relative to the rotation around the sense of the vector of $\vec{u}$

Therefore vector grad p must be directed along to one direct/straight with vector aaaa. In fact, let us assume reverse/inverse, i.e., let vector grad p composes certain angle with the sense of the vector of $\vec{u}$ If we turn the selected arbitrary coordinate system relative to the sense of the vector of $\vec{u}$ in certain angle, then neither vector of the aaaa nor of any another of the determining parameters will not change. Consequently, must not change vector grad p, which depends only on these parameters. But if grad p composes certain angle with the sense of the vector of $\vec{u}$ that during rotation its direction of relatively coordinate axes compulsorily it will change.

Page 8.

Hence it follows that vector grad p can be directed only in the

direction of the vector of $\vec{u}$ so that

$$\operatorname{grad} p = -c\vec{u}, \qquad (1.2.1)$$

where c is certain scalar quantity, which depends on the module/modulus of velocity vector u, and also values d, m, ρ, μ.

Let us examine first such filtration motions for which are unessential the force of inertia. To the number of similar inertia-free motions, belongs, on the strength of their extreme slowness, the majority of the filtration motions, which are encountered in practice [1].

FOOTNOTE [1]. So, during the development of the petrolium deposits of the rate of filtration in basic part of the layer they compose value of approximately 0.005 cm/s and less. ENDFOOTNOTE.

In this case the density ρ, which characterizes the inertia

properties of liquid, is unessential and is eliminated from the number of determining parameters. Thus, during inertia-free motions value c depends only on u, d, m and μ. Let us extract the dimensionality of the which interest us values:

$$[c]=\frac{M}{L^3T};\quad [u]=\frac{L}{T};\quad [d]=L;\ [\mu]=\frac{M}{LT};\ [m]=1. \qquad (I.2.2)$$

. From five values (I.2.2) it is possible to select three with independent dimensionality (for example u, μ and d). Then, according to π- theorem, dimensional analysis the unknown dependence will connect two dimensionless combinations of the indicated values. As one of the dimensionless quantities, it is convenient to take porosity m, as another let us select cd^2/μ. Thus, we have

$$cd^2/\mu=f(m),\quad c=\mu d^{-2}f(m). \qquad (I.2.3)$$

. After this the equation (I.2.1) can be presented in the form:

$$\operatorname{grad} p = -\mu d^{-2} f(m) \vec{u} \quad \text{or} \quad \vec{u} = -\frac{k}{\mu} \operatorname{grad} p; \; k = \frac{d^2}{f(m)}. \qquad (1.2.4)$$

This relationship/ratio is called the law of the filtration of darcys (from the name of the French scientist, who establish/installed him it is experimental in 1856). Value $k = d^2/f(m)$, introduced by equation (1.2.4), bears the name of permeability. Permeability has a dimensionality of area; it does not depend on the properties of liquid and is purely the dimensional characteristics of the porous medium.

In the physical system of ones, the permeability is measured in cm^2. However, the permeability of the majority of minerals is expressed in this case by very small numbers. So, the permeability coarse-grained is sandstone it composes 10^{-8}-10^{-9} cm^2; the permeability dense is sandstone - about 10^{-10} cm^2. In view of this in oil-field practice, will win acceptance one of permeability 1 d (darcy) = $1.02 \cdot 10^{-8}$ cm^2.

Page 9.

In the practice of hydraulic engineering calculations instead of the pressure, usually is utilized pressure head $H = p/\rho g$, and the law of darcys is record/written in the form:

$$\vec{u} = -C \operatorname{grad} H. \qquad (I.2.5)$$

. Value C, the having dimensionality of rate, is called filtration factor.

Recall that function f in expression (I.2.3) depends not only on porosity, but also on other dimensionless characteristics of the geometry of pore space. Were made the numerous attempts to present permeability as the function of porosity and significant dimension for the typical porous media both by means of the examination of the simplest models and by processing experimental data. These questions are minutely examined in book [71]. All the obtained results bear the particular character and have the narrow field of applicability. The

greatest reputation from the formulas of this kind uses the equation of Kozeni-Karmann, obtained on the basis of the analogy between the porous medium and the system of parallel tubes, which expresses permeability through the specific surface area of Σ and porosity m:

$$k = \frac{Km^3}{\Sigma^2}. \qquad (I.2.6)$$

Constant K is determined from experiment and proves to be different for the porous media different structure. Formula (I.2.6) is utilized mainly during calculations of the filtration resistance of the artificial porous media, used in chemical apparatuses; it they use also during the determination of the specific surface area of powders.

As can be seen from the given conclusion/derivation, the law of darcys is the consequence of hypothesis about the noninertia of the motion of liquid. The filtration flow, which follows to the law of darcys, is a special case of the creeping flow (widely known example of the creeping flow is the Stokes flow about the sphere). The flows

of such type are characterized by the predominance of viscous forces on inertia, i.e., very small Reynolds numbers (Re << 1). Therefore are represented unsuitable the numerous attempts to obtain the law of darcys by means of the averaging of the equations of nav'ye - Stokes. It is clear that any such a conclusion will be reduced in the final analysis to the attempt to calculate permeability according to the known geometric structure of the porous medium.

The law of darcys has very wide field of application/appendix and on its basis are obtained the basic results of filtration theory. There exist, however, the cases when the linear law of filtration darcys is not applicable. These cases, necessary the generalization of the law of darcys and the appearing in this case nonlinear problems of filtration theory will be examined below (chapter VIII). Now we will count all the motions of the obeying the law darcy in question.

Page 10.

Until now, it is assumed that porous medium was isotropic. If the porous medium is not isotropic, then of the common/general/total

considerations it is possible to assert that in the arbitrary orthogonal Cartesian system of the coordinates of x_1, x_2, x_3 the components of vector grad p are expressed as the components of the of the vector of $\vec{u}$ as follows [1]:

$$\frac{\partial p}{\partial x_i} = -c_{i\alpha}u_\alpha. \qquad (1.2.7)$$

where the c_{ij} are certain tensor.

FOOTNOTE [1]. Here and throughout we will assume to be summation over all values of the being repeated Greek indices, so that, for example $c_{i\alpha}u_\alpha$ means $c_{i1}u_1 + c_{i2}u_2 + c_{i3}u_3$. ENDFOOTNOTE.

In the case of inertia-free motions, the components of the tensor of c_{ij} can depend only on the ductility/toughness/viscosity of liquid μ, of those or other dimensional characteristics of the porous medium and module/modulus of the velocity vector of the filtration of $\vec{u}$

Analogous with the derivation of formula (I.2.7) it is possible to show that the $c_{ij} = \mu r_{ij}$, where the tensor of r_{ij} depends only on the dimensional characteristics of the porous medium and is called the tensor of specific filtration resistance; the components of the tensor of r_{ij} have a dimensionality of reverse/inverse area. Expressing, on the contrary, the components of velocity vector through the components of the vector of pressure gradient, we obtain

$$u_i = -\frac{k_{i\alpha}}{\mu}\frac{\partial p}{\partial x_\alpha}, \qquad (1.2.8)$$

where the tensor of K_{ij} it is reverse/inverse to the tensor of r_{ij} also it depends only on the dimensional characteristics of the porous medium, it has a dimensionality of area and is called the tensor of permeability. This dependence is the law of darcys for the anisotropic porous medium.

Let us show now that the tensor of the resistance of r_{ij} and the tensor of the permeability of K_{ij} are symmetrical, i.e., $r_{ij} = r_{ji}, k_{ij} = k_{ji}$. In fact, to the porous medium from the side of the filtering liquid acts the volume force, proportional to pressure

gradient; dimensionless proportionality factor depends only on the dimensional characteristics of the porous medium. The specific work of this force, i.e., work for time unit per unit volume of liquid - porous medium system, equal to the specific dissipation of energy by liquid in the porous medium, is equal to scalar product

$$\left(\operatorname{grad} p, \vec{u}\right) = \frac{\partial p}{\partial x_\beta} u_\beta = -\mu r_{\beta\alpha} u_\beta u_\alpha. \qquad (1.2.9)$$

It is obvious that the specific work of the forces of the interaction of liquid with the porous medium must not depend on the selection of the axes of coordinates x_1, x_2, x_3. But in order that the quadratic form of $r_{\beta\alpha} u_\beta u_\alpha$, proportional to this specific work, does not depend on the selection of the coordinate system, necessary and sufficiently, in order to $r_{\alpha\beta} = r_{\beta\alpha}$. Analogously it is possible to show that $k_{\alpha\beta} = k_{\beta\alpha}$.

In appendices the special role plays the anisotropy of the natural porous media, connected with sludging. In this case of permeability along layers, they have one value, while in perpendicular direction - another, usually considerably less.

Therefore one of the principal axes of the tensor of permeability - x_3 is perpendicular to the plane of stratification, and two others - x_1 and x_2 can be selected arbitrarily in the plane of stratification. System x_1, x_2, x_3 will be the main system at each point of the porous medium; in this system we have

$$k_{11}=k_{22}=k; \quad k_{33}=k_0; \quad k_{12}=k_{21}=k_{32}=k_{23}=k_{31}=k_{13}=0. \qquad (I.2.10)$$

Darcy's law in the selected coordinate system is record/written on the strength of relationship/ratios (I.2.10) as follows:

$$u_1=-\frac{k}{\mu}\frac{\partial p}{\partial x_1}; \quad u_2=-\frac{k}{\mu}\frac{\partial p}{\partial x_2}; \quad u_3=-\frac{k_0}{\mu}\frac{\partial p}{\partial x_3}. \qquad (I.2.11)$$

end section.

MT/ST-76-1860

Page 11.

§ 3. Dependence of the parameters of the liquid and porous medium on pressure.

Since the motion of liquid in the porous medium is caused the pressure differential, the final formulation of the majority of the problems of filtration theory entails the composition of differential equations for pressure distribution and in the establishment of the

corresponding initial and boundary conditions. Both during the composition of these equations and during their solution it is necessary to know, as depend on pressure the characteristics of the porous medium and its saturating liquid.

1. Let us examine first of all the effect of pressure on the properties of liquid - density ρ and ductility/toughness/viscosity μ.

For true liquids - the water and petroleum - density change are usually small. The being encountered in filtration motions pressure differentials (dozen kgf/cm²) are very small in comparison with the bulk moduli of the K_ρ of true liquids ($5 \cdot 10^3 - 2 \cdot 10^4$ kgf/cm²). Therefore for application/appendices it suffices to be restricted to linear dependence

$$\rho(p) = \rho_0 \left(1 + \frac{p - p_0}{K_\rho}\right). \qquad (I.3.1)$$

It follows, however, to keep in mind that although the compressibility of true liquids is small, it plays the considerable

role when compressive disturbances seize vast regions (here substantially the fact that the petroleum deposits usually border on stratal water, total volume of which is considerably greater than the volume of oil in deposit; as a result of this, the expansion of water during a decompression can completely compensate for the extracted volume of oil). The dependence of the viscosity of true liquids on pressure during pressure change in the same limits can be usually disregarded [1].

FOOTNOTE [1]. The aforesaid is not related to oil, which is located in contact with natural gas. In this case during a pressure increase, increases the amount of dissolved in oil gas, and its viscosity noticeably falls. ENDFOOTNOTE.

The filtration flows of gas are characterized by the fact that during their investigation, on one hand, almost always it is possible to disregard temperature changes, by considering them small, and with another, fact that in view of the large absolute values of pressure and jump/drops to consider gas ideal possible only with large tension. The equation of state of gas usually they record/write in the form:

$$\rho=\frac{p}{z\,(p,\,T)\,RT}\,. \qquad (1.3.2)$$

The advantages of this notation are connected with the fact that for function x (p, T), called the coefficient of supercompressibility, are comprised the tables and the curve/graphs, which cover a series of the procedurally important cases, and there are simple methods of its approximation calculus for gas mixture [27].

Page 12.

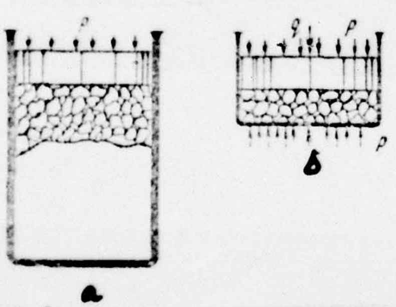


Fig I.3

The temperature in this equation usually can be considered constant and considered as parameter. Deviation z from unity (gas from ideality) is more considerable for heavier hydrocarbon gases.

According to elementary kinetic theory of gases, the viscosity of gas must not depend on pressure. This affirmation also not is applicable to the conditions, characteristic for a gas layer. At the fixed temperature the viscosity of gas it can change by dozens percent during a change in the pressure on dozens atmospheres.

2. Let us examine now question of how depend on pressure of liquid the properties of the porous medium - its porosity m and permeability k. Both these values characterize the structure of pore space, and their change in any point it is determined by the pressure of liquid and by stress tensor, which act in the skeleton of the porous medium. In this case, it should be noted that in experiments is determined their dependence not on the actual stresses, which act in skeleton, but on certain of their part which we will name fictitious stresses. For the explanation of this fact, let us disassemble the following elementary diagram of experiment. Let (Fig. I.3a) in cylindrical container with cross-sectional area, equal to unity, is be located certain volume of the porous medium, in which is

contained the liquid under pressure p. On the face side of this volume, lie/rests the impenetrable piston, through another side of which is located the liquid under that pressure p. On the strength of the known principle of hydrostatics - the principle of consolidation - this system is found in equilibrium state. For the explanation of the dependence of porosity on load, let us add to piston increment load q. Let us compute the compressive normal stress, which acts in the section of the volume of the porous medium by plane, to parallel piston; for this let us compose the equation of the equilibrium of part of the volume in question, limited by piston and plane of cross-section (Fig. I.3b). Disregarding frictional forces against the walls of the accomodating container and the dead weight of the medium and liquid, we obtain

$$\sigma + mp = q + p; \quad \sigma = q + p(1 - m), \qquad \text{(I.3.3)}$$

where σ - the actual stress, which acts in the porous medium (taking into account the unit of area of common/general/total section) and, obviously, not equal to the applied load q.

Page 13.

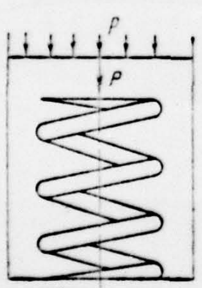


Fig. I.4.

A change of the porosity in pressure dependence with the fixed load as a whole little essential, is considered separately (this change it is caused by the compressibility of the material of the grains, constituting the porous medium which is small comparatively with the compressibility of the porous medium as a whole, since a change in the porosity occurs in essence because of the more tight packing of grains and only in very small measure - because of their compression; if we not at all consider the compressibility of the material of the grains, constituting the porous medium, then porosity with the fixed load will not depend on the pressure of liquid). It is possible to show also which with the fixed stresses σ a change in the pressure of liquid not at all will lead to a change in the volume of skeleton, regardless of the fact is such the compressibility of its material. Thus, the experiment in question gives to us the dependence of porosity on load q, by the component only the part of the actual stresses, which act in the skeleton of the porous medium:

$$q = \sigma^f = \sigma - p(1-m). \qquad (1.3.4)$$

. The value of σ^f we will subsequently call the fictitious stress.

The important special feature/peculiarity of the porous medium, noted above, entails the fact that changes in the occupied with it volume can occur during very small changes in their own volume of solid skeleton, almost exclusively because of its rearrangement. As the simplest model of a similar system can serve the spring, immersed in water (Fig. I.4). The volume of the cylindrical body, limited by spring, virtually does not change during a change in the pressure of liquid and can strongly change, if we apply along end/leads opposed forces. In formula for the calculation of spring compression, one should substitute the value of actual stresses minus of term, caused by the pressure of liquid.

Analogous considerations are used in the more general cases. Thus, the experiment, placed under conditions of arbitrary loading, will give to us the dependence of porosity not on the tensor of the actual stresses, which act in the skeleton of the porous medium, but from the tensor of fictitious stresses. In view of the fact that during action on porous medium one of hydrostatic pressure shearing stresses in the porous medium do not appear, the tangential components of the tensor of actual stresses and tensor of fictitious

stresses coincide, but normal components differ by value p (1 - m), have

$$\sigma^{f}_{ij} = \sigma^{v}_{ij} - p(1-m)\delta_{ij} \qquad (i, j = 1, 2, 3\ldots), \qquad (I.3.5)$$

where an σ^{f}_{ij} - the components of the tensor of fictitious stresses; σ^{v}_{ij} - the components of the tensor of actual stresses; the $\delta_{ij} = 1$ with cf $j = i, \delta_{ij} = 0$ with i $\neq$ j.

While values scalar, porosity and permeability can depend only on the invariants of the tensor of fictiticus stresses.

Page 14.

Following N. M. Gersevanov [37], their dependence on the second and third invariants of the tensor of fictitious stresses they disregard 1, whence

$$m = m(\Theta, p);\ k = k(\Theta, p);\ \Theta = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} = \frac{\sigma'_1 + \sigma'_2 + \sigma'_3}{3}, \qquad (I.3.6)$$

where the $\sigma'_1, \sigma'_2, \sigma'_3$ are the main normal fictitious stresses, and Θ are the average stress.

FOOTNOTE [1]. The possibility of this neglect is connected with the fact that in actuality the porosity, permeability etc. depend on the ratios of stresses to to certain constant for the medium values of the type of the coefficients of compressibility which it is not less than by an order more than stresses. Therefore neglect the second and third invariants means in reality neglect quadratic and cubic terms, i.e., linearization. ENDFOOTNOTE.

The value of Θ can be connected with pressure p, if we

examine the stressed state in layer. Let H be a depth of the occurrence of layer, h - its power, and ρ_0 is the average density of minerals. Usually the oil strata arrange/located on considerable depth under topographic surface and their power is small in comparison with the depth of occurrence, i.e. h << H. In this case it is possible to connect a change in the value of aaaa with a change in pressure p. In fact, the lying/horizontal above the layer minerals are supported by the skeleton of layer and by the which saturates layer liquid, so that the weight of the superincumbent minerals is balanced by the system of the stresses in the porous medium and by hydrodynamic pressure, liquid. Constituting layer system liquid - the porous medium can be visualized as certain deformed system, shearing stresses in which coincide with shearing stresses in the porous medium, and normal stresses are equal to the sum of the true normal stresses, which act in the porous medium, and the fractions normal obviously, to the product of porosity and the pressure of liquid). We have, thus, expression for the component of the sum voltage of σ_{ij}:

$$\sigma_{ij}=\sigma^{*}_{ij}+mp\delta_{ij}=\sigma^{f}_{ij}+(1-m)\,p\delta_{ij}+mp\delta_{ij}=\sigma^{f}_{ij}+p\delta_{ij}. \quad (1.3.7)$$

Let ρ be the total density of system liquid - the porous

medium, and g_i - the component of the G-vector of the force along the axis of x_i. Then the equation of the equilibrium of system liquid - the porous medium takes the form:

$$\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha} + \rho g_i = \frac{\partial \sigma^f_{i\alpha}}{\partial x_\alpha} + \frac{\partial p}{\partial x_i} + \rho g_i = 0. \qquad (I.3.8)$$

By considering liquid slightly compressible, it is possible to place in the equation (I.3.8) of the $\rho = \rho_*$, where of the ρ_* - the constant original value of the total density.

Page 15.

Thus, the total equation of the equilibrium of system liquid, the porous medium, finally is record/written in the form:

$$\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha} + \rho_* g_i = 0 \qquad (I.3.9)$$

and, as is evident, this equation does not depend on time. Let us show now that and sum voltages on roofing and bottom of layer (i.e. on upper and lower limiting layer surfaces) it is possible with a high degree of accuracy to consider constants. Physically the explanation of this fact is reduced to the following: the elastic displacement, caused by a change in the pressure of the liquid, saturating the rock/species of layer, proportional, obviously, to thickness of layer, it is distributed to entire enormous thickness H of the superincumbent array of minerals, so that the corresponding relative strains in this array are small and, therefore, low the appearing in it secondary stresses, in particular secondary stresses on roofing and the bottom of layer.

Let us explain this in somewhat more detail. Let us assume that the pressure of the liquid, which saturates layer, changed in comparison with the initial torque/moment on value δp. Let us

designate the value of a change of the pressure of liquid in that place where it is maximal, through the $\delta p_{макс}$. For maintaining the superincumbent minerals, it is necessary that the stress in the skeleton of the porous medium within layer would change also on the value of order δp. The corresponding relative strain in layer composed the value of order δp/E, where E is certain effective Young's modulus system, and the complete vertical displacement of point, for example the roofing of layer, the value of order v = hδp/E, where h is a thickness of layer. Let us note now that, after fastening the points of floating surface, i.e., after ensuring on floating surface equality to zero of elastic displacement, and having also replaced in all points of layer δp by $\delta p_{макс}$, we can only increase the appearing secondary stresses. Thus, if on the floating surface of the superincumbent array displacement equal to zero, and at depth H it has a value of the order of $v_{макс} = h\delta p_{макс}/E$, that, obviously, corresponding stress of $\sigma_{макс}$ has a value of the order of $\sigma_{макс} = v_{макс}E/H$. The ratio of this secondary stress to acting at depth H to the vertical compression stress [1], which is of the order $\rho_0 gH$ (ρ_0 - the average density of minerals is the value, approximately equal to 2.5 g/cm³), is equal by order of value

$$\frac{\delta p_{\text{макс}}}{\rho_0 g H} \sim \frac{h}{H}. \qquad (1.3.10)$$

FOOTNOTE [1]. If layer is inclined, then instead of the vertical normal stress one should take the stress, perpendicular to the direction of the stratification which usually has the same order. ENDFOOTNOTE.

The value of $\delta p_{\text{макс}}/\rho_0 g H$ usually does not exceed one-two the tenth; value h/H is vanishingly small, so that a change in the stress in all superincumbent arrays, in particular, on its boundaries is small in comparison with the initial stress.

Page 16.

Therefore it is possible to consider that during a change of the pressure of liquid in layer the stresses, which act on roofing and bottom of layer, remain constants.

The preceding/previous reasoning is substantially based on what Young's modulus the systems liquid - the porous medium E the module/modulus of the superincumbent array of minerals E_1 have identical order of magnitude (which usually occurs in actuality). If these Young's modulus strongly differed between themselves, then the expression (I.3.10) would contain cofactor E_1/E and with $E_1 >> E$ the relation of stresses could and not be small. Physically this means that in the case when the superincumbent thickness is accumulated from very rigid rock/species, they can be formed the arch/summaries, and during a change in the pressure of liquid stresses on roofing and bottom of layer will be changed.

If we now disregard the effect of such boundaries of the region of filtration as walls of holes (these boundaries have comparatively very small extent; their effect will be estimated below), then of the independence from time of the equations of the equilibrium of system liquid - the porous medium (I.3.9) and stresses on roofing and bottom of layer follows the important conclusion/derivation about the

independence of the total stressed state in system liquid - the porous medium from time, so that

$$\frac{\partial\sigma_{ij}}{\partial t}=0,$$

whence

$$\frac{\partial(\sigma^f_{ij}+p\delta_{ij})}{\partial t}=0. \qquad (\text{I}.3.11)$$

Coagulating equations (I.3.11) (i.e. set/assuming i, j = 1, 2, 3 and summarizing the being obtained equations), we have

$$\frac{\partial}{\partial t}(\sigma^f_{11}+\sigma^f_{22}+\sigma^f_{33}+3p)=0,$$

whence it escape/ensues important relationship/ratio

$$\frac{\partial(\Theta+p)}{\partial t}=0,\quad \frac{\partial\Theta}{\partial t}=-\frac{\partial p}{\partial t}. \qquad (\text{I}.3.12)$$

. Page 17.

Chapter II.

BASIC PROBLEMS OF UNSTEADY FILTRATION.

§ 1. Equation of continuity.

Let us examine the balance of the mass of liquid in the arbitrary element of volume of the porous medium of V, limited by surface of S. For infinitesimal time dt, the inflow of liquid inside cell/element is equal according to the determination of rate of filtration

$$-dt\int_S \rho u_n d\sigma = -dt\int_S \rho(\vec{u}\vec{n})\,d\sigma \qquad (II.1.1)$$

($\vec{n}$ - the unit vector of standard; as the positive direction of standard is accepted the direction of external normal to the surface; u_n are normal to the surface a velocity component of filtration). An increase in the mass of liquid within this cell/element is equal to

$$\left(\frac{\partial}{\partial t}\int_V m\rho dv\right)dt = \left(\int_V \frac{\partial m\rho}{\partial t}dv\right)dt. \qquad (II.1.2)$$

Equating expressions (II.1.1) and (II.1.2) and utilizing a formula of the transformation of surface integral into volumetric

$$\int_S \rho u_n d\sigma = \int_V \operatorname{div}(\rho\vec{u})\, dv,$$

we find

$$\int_V \left(\frac{\partial m\rho}{\partial t} + \operatorname{div}\rho\vec{u}\right) dv = 0,$$

whence on the strength of the arbitrariness of cell/element of V it escape/ensues the equation of continuity

$$\frac{\partial m\rho}{\partial t} + \operatorname{div}\rho\vec{u} + 0. \qquad (II.1.3)$$

Page 18.

§ 2. Elastic mode/conditions of filtration.

1. The quite simple and most studied case of unsteady filtration is the filtration of slightly-compressible liquid in elastic-deformed layer (in technical application/appendices these problems were called the name the problems of the elastic mode/conditions of filtration). As the basis of investigation, places the system of equations of the law of filtration and equation of the continuity:

$$\frac{\partial m\rho}{\partial t}+\operatorname{div}\rho\vec{u}=0;\quad \vec{u}=-\frac{k}{\mu}\operatorname{grad}p. \tag{II.2.1}$$

In order to obtain closed system of equations, it is necessary to use the fact that the property of liquid (density ρ and viscosity μ), just as the porosity and the permeability of the porous medium, are the functions of pressure (we assume to be motion isothermal).

On the strength of (I.3.12) we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t}, \quad \frac{\partial m}{\partial t} = \frac{\partial m}{\partial \Theta}\frac{\partial \Theta}{\partial t} + \frac{\partial m}{\partial p}\frac{\partial p}{\partial t} = \left(\frac{\partial m}{\partial p} - \frac{\partial m}{\partial \Theta}\right)\frac{\partial p}{\partial t} \ldots$$

On the basis of assumption about the weak compressibility of the liquid and porous medium, it is possible to consider relative changes of values ρ and $\propto$ small and the coefficients of p/ t in the preceding/previous formulas by constants:

$$\frac{\partial \rho}{\partial p} = \frac{\rho_0}{K_\rho}; \quad \frac{\partial m}{\partial p} - \frac{\partial m}{\partial \Theta} = \frac{m_0}{K_m}; \quad \frac{\partial (k/\mu)}{\partial p} = \frac{k_0}{\mu_0 K_\kappa}. \qquad (II.2.2)$$

Experimental data show that in the real cases

$$(p - p_0)/K_m \ll 1; \quad (p - p_0)/K_\rho \ll 1 \text{ и т. д.}$$

Substituting the second equation (II.2.1) in the first and converting the being obtained relationship/ratic taking into account (II.2.2), we find, disregarding low values,

$$m_0\rho_0\left(\frac{1}{K_\rho}+\frac{1}{K_m}\right)\frac{\partial p}{\partial t}+\frac{\rho_0 k_0}{\mu_0}\left[\nabla^2 p+\left(\frac{1}{K_\rho}+\frac{1}{K_m}+\frac{1}{K_k}\right)(\operatorname{grad} p)^2\right]=0.$$

If δp is a characteristic change in pressure, and L - reference length, then the first term in brackets has, obviously, an crder $\delta p/L^2$, and the second $(\delta p)^2/L^2K$. Hence it follows that the second term in the taken approach/approximaticn also one should disregard [1].

FOOTNOTE [1]. Comp. with note on page 14 about possibility to disregard

the dependence of porosity and permeability on the second and third invariants of stress tensor. ENDFOOTNOTE.

Thus, we have

$$\frac{\partial p}{\partial t} = \varkappa \Delta p = \varkappa \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right), \qquad (II.2.3)$$

where a coefficient

$$\varkappa = \frac{k_0}{\mu_0 m_0} \left(\frac{1}{K_m} + \frac{1}{K_p} \right)^{-1} \qquad (II.2.4)$$

is called the coefficient of piezoconductivity.

Page 19.

Equation (II.2.3) usually is called the equation of elastic mode/conditions or, for V. N. Shchelkacheva's proposition, by the equation of piezoconductivity. It coincides with the well known classical equation of thermal conductivity.

The formulation of the problem of the elastic mode/conditions of layer was given and the works of Theiss [160], Jacob [138] and independently by V. N. Shchelkachev [123].

2. Let us examine the formulation of the basic problems of the theory of elastic mode/conditions. Let us determine the distribution of pressure p in certain closed domain of space D in the extent/elongation of time interval $0 \leq t \leq T$. From the theory of the equation of thermal conductivity, it is known that if we assign on boundary g of domain D the linear combination of pressure and its derivative along the normal to boundary of the region

$$\alpha p + \beta \frac{\partial p}{\partial n}\Big|_{\Gamma} = f(x, y, z, t) \qquad \text{(II.2.5)}$$

and to assign the initial distribution of pressure in domain D

$$p(x, y, z, 0) = \varphi(x, y, z). \qquad \text{(II.2.6)}$$

that there is distribution of pressure p (x, y, z, t), and besides only, that satisfies an equation (II.2.3), continuous in closed domain D, including boundary, and satisfying the conditions (II.2.5) and (II.2.6).

The formulated problem covers almost all basic problems of the theory of the elastic mode/conditions of filtration.

Let us examine in more detail the physical sense of those or other supplementary conditions.

The domain in which searches for pressure distribution of liquid, usually is the porous layer, which partially has impenetrable boundaries, and which is partially communicated with other layers and

its revealing holes. On impenetrable boundaries must be satisfied the obvious condition of the absence of flow - the equality of the normal component of rate of filtration to zero:

$$u_n = 0,$$

whence, utilizing a law of darcys, we obtain

$$(\mathrm{grad}\, p)_n = \frac{\partial p}{\partial n} = 0. \qquad \text{(II.2.7)}$$

On the sections of boundary with the domains in which the redistributions of pressure in practice does not occur (the "domain of supply"), pressure can be considered as constant and known, so that

$$p|_\Gamma = f(x, y, z). \qquad \text{(II.2.8)}$$

This condition is correct, if, for example, the layer in question borders on the high-permeability domain, the supply of liquid in which is very great.

Page 20.

Pressure on the boundary of this domain closely to mean pressure in it and in view of its large volume barely depends on the processes, which proceed in the investigated domain. A characteristic example is the petroleum deposit, surrounded from all sides by vast water-bearing domain.

In the examination of unsteady processes in deposit pressure in water-bearing domain can be considered constant. It follows, however,

to distinctly visualize, that the concept of the range of constant pressure is not absolute. Than more prolonged character bear pressure changes, by the fact to large range they are spread.

Part of the boundary of the region of filtration is usually formed by the walls of hole or drainage galleries. On this part of the boundary, most frequently is assigned either the pressure of liquid or its flow through the walls of hole. The selection of one condition or the other depends on the operating mode of hole or gallery. There can be the more complex conditions when is assigned communication/connection of pressure with fluid flow rate. The assignment of fluid flow according to the law of darcys is equivalent to the assignment of normal derivative of pressure.

Conditions of this type are satisfied with those sections of the boundaries through which can occur the exchange of liquid with the adjacent layers through the comparatively slightly permeable cross connections. If the thickness of cross connection Δ is small, and pressure p' after it it is possible to consider constant, then the flow rate of the effluent through the section of the cross connection with an area of ds will compose $\frac{k'(p-p')\,ds}{\mu\Delta}$. . This amount of liquid must be equally

$$u_n ds = -\frac{k \partial p}{\mu \partial n} ds,$$

where of the u_n - the normal projection of rate of filtration on the section of boundary in question. Hence we have

$$\frac{\partial p}{\partial n} + \frac{k' p}{k \Delta} = \frac{k' p'}{k \Delta} = \text{const}, \qquad (II.2.9)$$

i.e. third-order conditions.

Everything three types of conditions are special cases of common/general/total condition (II.2.5). Thus, assigning the initial distribution of pressure and the indicated boundary conditions, we obtain the unambiguously solved problem.

§ 3. Equations of the free-flow filtration of the incompressible fluid.

1. General formulation of the problem. By free-flow filtration motion is understood the motion with the floating surface on which the pressure of liquid constantly is equal external atmospheric to pressure. Most frequently it is necessary to meet the free-flow motion of underground water; the free-flow motion of oil is encountered comparatively rarely, only with mine yield.

Page 21.

Let us examine free-flow motion in the homogeneous and isotropic porous medium, zone of flow we will assume to be bounded below impenetrable and by curved surface - by confining stratum.

The law of darcys in the case in question can be written in the form:

$$\vec{u} = -C \operatorname{grad} \bar{h}; \quad \bar{h} = z + p/\rho g. \qquad (II.3.1)$$

Value C, the having dimensionality of rate, is called the filtration factor, $\bar{h}$ - by pressure head, and the function of $C\bar{h}$ filtration potential. Let us note that for the free-flow motion of change the pressures usually so small, that the porous medium possible chounted, and the incompressible fluid, so that C = const, ρg = const.

In a precise setting the study of free-flow filtration motion presents exceptional difficulties of mathematical character; the relating here settings of problems and results can be found in the book by P. Ya. Polubarinova-Kochina [94]. Therefore it is necessary to turn to some simplified settings.

Great significance has the approximate formulation of the problem of free-flow filtration, which corresponds to the case of the

motion which we will call flat. By flat filtration motion is understood the motion, which proceeds in layers with the final depth of the confining stratum in which vertical the component of the rate of filtration of aaaa is small in comparison with horizontal component. Since the characteristic rate in free-flow filtration motion is the rate C, horizontal the component of rate of filtration can be either order C or small in comparison with C. In both cases it is clear that vertical the component of aaaa is small in comparison with C, i.e.,

$$u_z \ll C = \frac{k\rho g}{\mu}. \qquad \text{(II.3.2)}$$

This inequality can be rewritten still thus:

$$\frac{\mu u_z}{k} \ll \rho g. \qquad \text{(II.3.3)}$$

But $\frac{\mu u_z}{k}$ is that part vertical the components of pressure gradient, which is caused by the filtration of liquid. Inequality

(II.3.3) shows in such a way that vertical the component of filtration pressure gradient is small in comparison with hydrostatic pressure gradient. Therefore pressure distribution according to vertical line can be in the case of flat motions considered hydrostatic.

Let us derive important for further reasonings relationship/ratio. Let us examine volume by V, limited free surface of liquid and by certain cylindrical surface with vertical generatrices.

Page 22.

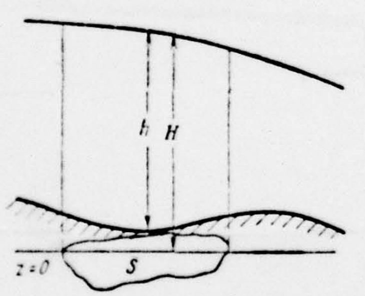


Fig. II.1.

Let us designate by h distance from free surface of liquid to confining stratum, and by H - the distance from floating surface to the horizontal plane z = 0 (Fig. II.1); it is obvious, $\delta h/\delta t = \delta H/\delta t$. The volume of the liquid, included in volume of V, is equal to

$$\int_S mhdS, \tag{II.3.4}$$

where area/site S it is the projection of volume on horizontal plane (see Fig. II.1). A change of the amount of liquid in volume of V for infinitesimal time interval dt is equal therefore

$$dt\left(\int_S m\frac{\partial H}{\partial t}dS\right) = dt\int_S m\frac{\partial h}{\partial t}dS. \tag{II.3.5}$$

At the same time this change is equal to the inflow of liquid into volume of V from without for time dt, equal to

$$-dt\int_{\gamma}dl\int_{H-h}^{H}u_n dz=-dt\int_{\gamma}w_n dl, \qquad (II.3.6)$$

where γ is the closed loop, which limits area/site S, and u_n are normal the component of the velocity of $\vec{u}$; w_n - normal the component of the vector of the flow of the $\vec{w}$ of that determined by relationship/ratio

$$\vec{w}=\int_{H-h}^{H}\vec{u}dz. \qquad (II.3.7)$$

Equating (II.3.5) and (II.3.6) and utilizing a formula of the transformation of contour integral into integral in terms of area

$$\int_{\gamma} w_n dl = \int_{S} \operatorname{div} \vec{w} dS,$$

we obtain

$$\int_{S} \left(m \frac{\partial h}{\partial t} + \operatorname{div} \vec{w} \right) ds = 0, \qquad \text{(II.3.8)}$$

whence, using the arbitrariness of area/site S, we find equation

$$m \frac{\partial h}{\partial t} + \operatorname{div} \vec{w} = 0. \qquad \text{(II.3.9)}$$

According to the law of darcys, the rate of filtration is determined by relationship/ratio (II.3.1).

Page 23.

Since, on preceding/previous, the pressure is distributed by vertical line with an accuracy to low values in hydrostatic law, the value of $\tilde{h}$ along each vertical line will be constant and equal to H:

$$\tilde{h}(x, y, z, t) = H(x, y, z, t) + O(u_z/C); \quad \vec{u} = -C \operatorname{grad} H + O(u_z).$$

Thus, the rate of $\vec{u}$ can be, by disregarding low values, removed from under the sign of integration for vertical line in the relationship/ratio (II.3.7), which determines the vector of $\vec{W}$. Then we obtain

$$\vec{w} = -Ch \operatorname{grad} H. \qquad (II.3.10)$$

Substituting (II.3.10) in (II.3.17), we have

$$\frac{\partial h}{\partial t} = \frac{C}{m} \operatorname{div}(h \operatorname{grad} H). \qquad (II.3.11)$$

In this equation one should substitute relationship/ratio

$$H(x, y, t) = h(x, y, t) + h_0(x, y),$$

determine the vertical coordinate of floating surface H through its distance h of confining stratum and distance h_0 from confining stratum to reference plane z = 0; we will obtain final equation for

determining of h. Specifically, if the surface of confining stratum is horizontal plane, then it it is possible to accept for reference plane and, therefore, h_0 (x, y) it is possible to consider equal to zero. Then H = h, and equation (II.3.11) assumes the form:

$$\frac{\partial h}{\partial t} = a\Delta h^2; \qquad a = \frac{C}{2m} = \frac{k\rho g}{2m\mu}. \qquad \text{(II.3.12)}$$

equations (II.3.11) and (II.3.12) were given by Boussinesq [133].

§ 4. Fundamental equations of the filtration of gas.

During the study of the filtration of gas, the primary meaning has the fact that the compressibility of gas usually on several orders exceeds the compressibility of the porous medium. Taking this into account of fact in the equation of continuity

$$\frac{\partial m\rho}{\partial t} + \operatorname{div} \rho \vec{u} = 0 \qquad \text{(II.4.1)}$$

a change of porosity m in time can be disregarded, so that we will obtain

$$m \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \vec{u} = 0. \qquad \text{(II.4.2)}$$

For oho in order to obtain the locked system of equations, it is again necessary to utilize communication/connection of the density of gas ρ with its pressure p temperature of T:

$$\rho = \rho(p, T), \qquad \text{(II.4.3)}$$

therefore in problem appears new alternating/variable T, and for the closing/shorting of system of equations necessary to add one additional equation - the equation of energy.

Page 24.

However, if in the medium there are no sources of isolation or energy absorption, then changes in the temperature in the process of the flow of gas are extremely small, and during the calculation of the pressure field of gas them can be disregarded. This fact it is easy to understand, if we consider, in the first place, the extreme smallness of rate of filtration and, in the second place, the presence of thermal ballast - the skeleton of the porous medium, effectively suppressing change in the temperature. We will be therefore consider that

$$\rho=\rho(p, T_0)=\rho(p), \qquad \text{(II.4.4)}$$

where T_0 is constant temperature.

Connecting up equations (II.4.2) and (II.4.4) the equation of the law of filtration (assumed to be linear)

$$\vec{u} = -\frac{k}{\mu}\,\mathrm{grad}\,p, \qquad \text{(II.4.5)}$$

we obtain the locked system of equations. Eliminating rate of filtration, we have

$$m\frac{\partial\rho}{\partial t} = k\,\mathrm{div}\left(\frac{\rho}{\mu}\,\mathrm{grad}\,p\right). \qquad \text{(II.4.6)}$$

In equation (II.4.6) ρ is a known function of pressure. It is analogous and the viscosity of gas, which depends in the general case on pressure and temperature, it can represented in the form:

$$\mu = \mu(p, T_0) = \mu(p). \qquad \text{(II.4.7)}$$

. Thus, and viscosity can be considered the known function only of pressure.

Let us introduce now functions

$$P(p) = k\int_0^p \frac{\rho(p)\,dp}{\mu(p)}\,;\ \varkappa(P) = m\left(\frac{d\rho}{dP}\right)^{-1}. \qquad \text{(II.4.8)}$$

Equation (II.4.6) accepts in this case the form:

$$\frac{\partial P}{\partial t} = \varkappa(P)\,\Delta P. \qquad \text{(II.4.9)}$$

It is possible to show that the equation for a pressure will preserve form (II.4.9), also, if is considered the deformability of the porous medium, i.e., pressure dependence of porosity and permeability (medium is considered as before uniform).

Page 25.

In the simplest case when gas can be considered thermodynamically ideal, with the viscosity, which does not depend on pressure,

$$\mu = \text{const}, \quad \rho = \frac{\rho_0 p}{p_0} \qquad \text{(II.4.10)}$$

(p_0 and ρ_0 are constants). In this case,

$$P(p) = \frac{k\rho_0 p^2}{2\mu p_0}; \quad \varkappa = \frac{kp}{m\mu}, \qquad \text{(II.4.11)}$$

and the equation (II.4.9) is converted to the form:

$$\frac{\partial p^2}{\partial t} = \frac{kp}{m\mu} \Delta p^2 \qquad \text{(II.4.12)}$$

or

$$\frac{\partial p}{\partial t} = \frac{k}{2m\mu} \Delta p^2. \qquad \text{(II.4.13)}$$

Equations (II.4.12) and (II.4.13) are derived under the assumption the temperature constancies of gas T_0. Therefore them usually they call by the equations of the isothermal filtration of gas.

Equation (II.4.13) - basic for the theory of the filtration of gas - is obtained for the first time by L. S. Leybenzon [70], and then, somewhat later, in the work of Muskat and Botset [148]. Transformation (II.4.8) also rises from its from the works of L. S. Leybenzon. Therefore we will call function P (p) the function of Leybenzon. Further equation (II.4.13) coincides with the equation of Boussinesq (II.3.12) for a pressure head during flat free-flow filtration motions. This analogy, for the first time discovered by L. S. Leybenzon, makes it possible to examine the study of the isothermal filtration of gas and flat free-flow motions of the incompressible fluid as one problem.

Pages 26-56.

Chapter III.

THE THEORY OF THE ELASTIC [redacted]/CONDITIONS OF FILTRATION.

§1. One-dimensional rectilinear-parallel motion.

By the elastic mode/conditions of filtration as already it was mentioned above, is understood the filtration of the elastic slightly-compressible liquid in the elastic porous medium. In these conditions pressure distribution is described by the classical equation of thermal conductivity (II.2.3). The detailed technique of the solution to this equation in different initial and boundary conditions is used also to the problems of the theory of elastic mode/conditions. Diverse concrete/specific/actual solutions can be borrowed, for example, from management/manual to Carslow and Jaeger [54] and from other sources. however, the problems of filtration theory have their specific character, connected with the presence of certain low parameters (for example the ratio of a radius of the hole to the size/dimension of layer), which in a number of cases substantially simplifies solutions. Therefore the given examples are intended not only to illustrate setting and the methods of the solution of the basic problems, but also to focus attention on this specific character, which differs these problems from the problems of thermal conductivity.

The reader, interested in knowledge with other aspects of the theory of elastic mode/conditions, can revert to books [124, 125, 3,

38], where examined large number of problems, from the quite simple to the very complex problems of motion in heterogeneous layers.

1. Let us examine the motions of liquids, for which the rate is parallel to X-axis and does not depend on coordinates y and z. Pressure in this case satisfies an equation

$$\frac{\partial p}{\partial t} = \varkappa \frac{\partial^2 p}{\partial x^2}, \quad 0 \leqslant x \leqslant L. \qquad \text{(III.1.1)}$$

Are most interesting the cases for which at the moment the motion in layer is stationary. Since the stationary distribution of pressure also satisfies an equation (III.1.1), is convenient to count off pressure at each point from steady-state value p_0 (x).

Page 27.

Introduced thus difference $P = p - p_0$ satisfies an equation (III.1.1) with zero initial condition

$$P(x, 0) = 0. \qquad \text{(III.1.2)}$$

Let us assume that in plane x = L the pressure retains constant value equal to initial:

$$P(L, t) = 0. \qquad \text{(III.1.3)}$$

This condition is satisfied, if, for example, the range in question borders on the vast well penetrated aquifer. Let us designate by F (t) the function, which describes pressure change in the initial section x = 0. In order to obtain the solution to equation (III.1.1) in the indicated initial and boundary conditions, let us apply to it the Laplace transform [42, 43, 63]:

$$L\{P(x, t)\} = \overline{P}(x, \sigma) = \int_0^\infty e^{-t\sigma} P(x, t)\, dt, \qquad \text{(III.1.4)}$$

as a result of which for transforms $\overline{P}$ we will obtain equation

$$\frac{d^2\overline{P}}{dx^2} - \frac{\sigma}{\varkappa}\overline{P} = 0 \qquad \text{(III.1.5)}$$

under boundary conditions

$$\overline{P}(0) = L\{f(t)\} = F(\sigma), \quad \overline{P}(L) = 0. \qquad \text{(III.1.6)}$$

The unknown solution to equation (III.1.5) takes the form:

$$\overline{P} = F(\sigma)\frac{\operatorname{sh}[(L-x)\sqrt{\sigma/\varkappa}]}{\operatorname{sh}(L\sqrt{\sigma/\varkappa})}. \qquad \text{(III.1.7)}$$

. Besides pressure distribution in layer for application/appendices, it is usually important to know also fluid flow through the initial section of layer x = 0 and through the moved away boundary (current generator L).

The flow rate of liquid, which is necessary to the unit of area, is equal to

$$Q(x, t) = -\frac{k}{\mu}\frac{\partial P}{\partial x}. \qquad \text{(III.1.8)}$$

. Utilizing (III.1.7), we obtain for transforms Q of expression

$$\overline{Q}(0) = -\frac{k}{\mu}\sqrt{\frac{\sigma}{\varkappa}}F(\sigma)\,\mathrm{cth}\left(L\sqrt{\frac{\sigma}{\varkappa}}\right);$$

$$\overline{Q}(L) = \frac{k}{\mu}F(\sigma)\sqrt{\frac{\sigma}{\varkappa}}\,\frac{1}{\mathrm{sh}\left(L\sqrt{\frac{\sigma}{\varkappa}}\right)}. \qquad \text{(III.1.9)}$$

. Page 28.

Relationship/ratios (III.1.7) and (III.1.9) give the solution to stated problem, if we use common/general/total inversion formula for the Laplace transform:

$$P(x, t) = \frac{1}{2\pi i}\int\limits_{\gamma - i\infty}^{\gamma + i\infty}\overline{P}(x, \sigma)\,e^{\sigma t}d\sigma, \ \gamma > 0. \qquad \text{(III.1.10)}$$

. 2. The application/use of this operating method to the problems of elastic mode/conditions is convenient in that relation,

which makes it possible easily to investigate the asymptotic behavior of the obtained solution in the large and low values of time, even without extracting it completely. This fact is especially important in connection with the complex problems for which the effective realization of reverse/inverse transform is hinder/hampered. Furthermore, it gives considerable simplifications during the solution of the reverse problem when we are speaking about determining the hydrodynamic characteristics of layer from measurement data of pressures and flow rates (comp. §4).

From the formulas of the Laplace transform (III.1.4) and (III.1.10) it is directly evident that the behavior of solution in the low values of time t is determined by the asymptotic behavior of image with large $|\sigma|$ and vice versa - the behavior of the converted function with small $|\sigma|$ determines the asymptotic behavior of original with large t.

The different reception/procedures of the determination of asymptotic behaviors and their foundation can be found in books [42, 63].

We define concretely now the form of the function f (t), after accepting it equal to the constant p^0. In this case, it is assumed that at the moment the pressure on the boundary of layer takes the new fixed value; with this $\bar{P} = p^0/\sigma$. From (III.1.7) and (III.1.9) we have

$$\overline{P}=\frac{p^0}{\sigma}\frac{\operatorname{sh}\left[(L-x)\sqrt{\frac{\sigma}{\varkappa}}\right]}{\operatorname{sh}\left(L\sqrt{\frac{\sigma}{\varkappa}}\right)};\ \overline{Q}(0)=-\frac{kp^0}{\mu\sigma}\sqrt{\frac{\sigma}{\varkappa}}\operatorname{cth}\left(L\sqrt{\frac{\sigma}{\varkappa}}\right). \tag{III.1.11}$$

Let us examine first the behavior of solutions in short times, i.e., we will examine (III.1.11) with large $|\sigma|$. Is expressed in (III.1.11) the hyperbolic functions in terms of exponential and, counting $2L\sqrt{\frac{\sigma}{\varkappa}}\gg 1$, is decomposed these expressions in power series $e^{-2L\sqrt{\frac{\sigma}{\varkappa}}}$. Then

$$\bar{P}(x,\sigma)=\frac{p^0}{\sigma}\left\{\sum_{n=0}^{\infty}\exp\left[-\sqrt{\frac{\sigma}{\varkappa}}(x+2Ln)\right]-\right.$$

$$\left.-\sum_{n=0}^{\infty}\exp\left[-\sqrt{\frac{\sigma}{\varkappa}}(2L(n+1)-x)\right]\right\};$$

$$\bar{Q}(0)=-\frac{p^0}{\sqrt{\varkappa\sigma}}\left\{\sum_{n=0}^{\infty}\left[\exp\left(-2Ln\sqrt{\frac{\sigma}{\varkappa}}\right)+\exp\left(-2L(n+1)\sqrt{\frac{\sigma}{\varkappa}}\right)\right]\right\}. \quad \text{(III.1.12)}$$

. Page 29.

Producing term-by-term reversion of series (III.1.12) (corresponding inversion formulas are derive/concluded, for example,

in the book of M. A. Lavrentyev and B. V. Shabat [63], chapter 6, §1,), we have

$$P(x, t) = p^0 \sum_{n=0}^{\infty} \left[\operatorname{erfc} \frac{2Ln + x}{2\sqrt{\varkappa t}} - \operatorname{erfc} \frac{2L(n+1) - x}{2\sqrt{\varkappa t}} \right]; \qquad (III.1.13)$$

$$Q(0) = -\frac{kp^0}{\mu\sqrt{\pi\varkappa t}} \left[1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{n^2L^2}{\varkappa t}\right) \right].$$

3. The obtained series, as it is easy to be convinced, converge with all t and x. We allow at first, that are examined short times and $\frac{L^2}{\varkappa t} \gg 1$. Then in expression for P (x, t) it is possible all the values erfc to replace them maximum erfc ∞ = with 0, with the exception of the term of a series of $\operatorname{erfc} \frac{x}{2\sqrt{\varkappa t}}$. Analogously in expression for Q (0) all terms of a series become zero. We have

$$P(x, t) = p^0 \operatorname{erfc} \frac{x}{2\sqrt{\varkappa t}}; \quad Q(0) = -\frac{kp^0}{\mu\sqrt{\pi\varkappa t}}. \qquad \text{(III.1.14)}$$

The obtained formulas make double sense. on one hand, they describe pressure distribution in the layer of the finite length L during short times of $\varkappa t \ll L^2$. on the other hand, they give pressure distribution in arbitrary point in time in the layer of the "infinite" extent $L \to \infty$. The fact is that the final (not infinitely small) change in the pressure is spread for preset time only to the final distance, and, if are examined short times, possible to consider layer infinite. The solution of problem for an infinite layer is self-similar: independent alternating/variable x and t enter in solution not separately, but only in the combination of $x/\sqrt{\varkappa t}$. The self-similarity of solution is simple result of the absence in the statement of the problem of constants, from which it is possible to form the values of the dimensionality of length or time. Self-similar solutions will be minutely examined below (chapter IV).

The equation of thermal conductivity (III.1.1) is linear and on the strength of this allow/assumes the superposition of solutions. This allows, utilizing the given solution for an abrupt change of the pressure in the initial section of layer, to construct the solution, which corresponds arbitrary boundary condition

$$P(0, t) = f(t) \qquad \text{(III.1.15)}$$

and which becomes zero in t = 0 and x = L (Duhamel integral):

$$P(x, t) = \int_0^t \frac{df(\tau)}{d\tau} P_1(x, t-\tau)\, d\tau. \qquad \text{(III.1.16)}$$

. Page 30.

Specifically, for an infinite layer

$$P(x,t)=\int_0^t \frac{df(\tau)}{d\tau}\operatorname{erfc}\frac{x}{2\varkappa\sqrt{t-\tau}}\,d\tau;\quad Q(0)=-\frac{k}{\mu\sqrt{\pi\varkappa}}\int_0^t \frac{df(\tau)}{d\tau}\frac{d\tau}{\sqrt{t-\tau}}. \tag{III.1.17}$$

If is assigned not pressure but the flow rate through the end section of infinite layer

$$Q(0)=q(t), \tag{III.1.18}$$

that solution as it is not difficult to show in that manner, it takes the form:

$$P(x,t)=-\frac{\mu}{k}\frac{\sqrt{\varkappa}}{\pi}\int_0^t q(t-\tau)\exp\left(-\frac{x^2}{4\varkappa\tau}\right)\frac{d\tau}{\sqrt{\tau}}. \tag{III.1.19}$$

4. Solution (III.1.14) it represents "thermal wave", which is

spread from torque/moment t = 0 of the point x = 0 in the positive direction of X-axis. Therefore expression (III.1.13) can be considered as result of the superposition of the thermal waves of the same amplitude p^0, which are spread from points x = -2Ln on the right and from points x = 2Ln to the left, beginning with the same point in time, the waves, which are spread to the left, having opposite sign. To this interpretation it is possible to give simple physical sense. Let us attempt to satisfy equation (III.1.1) and the placed boundary conditions with the aid of solutions of the type of thermal wave. It is obvious, all the such solutions they satisfy the initial conditions. In order to satisfy condition with x = L, let us add to solution (III.1.14) the thermal wave, emerging from point x = 2L in the negative direction of axis and which has the opposite sign:

$$f_1(x, t) = -p^0 \operatorname{erfc} \frac{2L - x}{2\sqrt{\varkappa t}}. \qquad \text{(III.1.20)}$$

On the strength of symmetry it is obvious that the total solution becomes zero with x = L. however, in x = 0, constructed solution no longer is equally accurate p^0. In order to compensate for the discrepancy, caused by the second thermal wave, let us add the third wave of the same sign and direction, as the is first, and

cutgoing from point x = 2L. In this case, appears the discrepancy under boundary condition with x = L, for compensation for which it is necessary to add backward wave from point x = the yawl etc. It is not difficult to see that in this way we will arrive at solution (III.1.13). Hence it is clear that thus far the values of time in question are not too great, sufficient to limit ourselves to the account of waves from several nearest to the point in question sources [this, of course, is easy to see and directly from (III.1.13)].

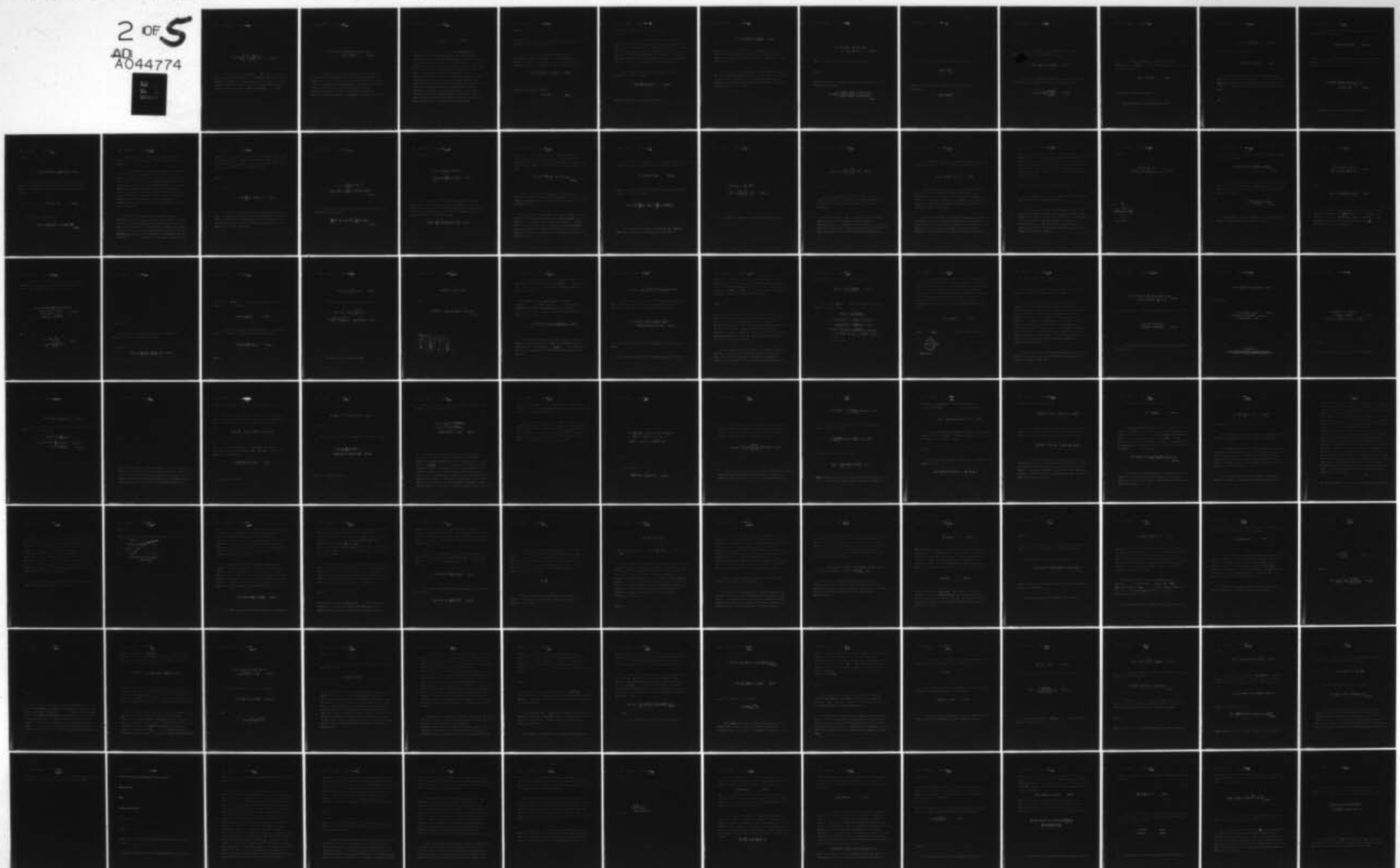
Page 31.

In this way it is possible to obtain simple expressions for a sufficiently prolonged initial cycle of motion.

5. Now let us examine the opposite asymptotic behavior: $t \gg \frac{L^2}{\varkappa}$. In this case the expression (III.1.13) is inconvenient to those that it is necessary to summarizemany terms of a series. In order to obtain solution in more convenient form, we will be turned again to relationship/ratio (III.1.7), set/assuming in it again f = p⁰. Utilizing an inversion formula (III.1.10), we have

AD-A044 774 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 13/11
THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS (TEORIY--ETC(U)
JAN 77 G I BARENBLATT, V M YENTOV, V M RYZHIK
UNCLASSIFIED FTD-ID(RS)T-1860-76-PT-1 NL

2 OF 5
AD
A044774



$$P_1(x,\ t)=\frac{p^0}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\operatorname{sh}\left[(L-x)\sqrt{\frac{\sigma}{\varkappa}}\right]}{\operatorname{sh}\left[L\sqrt{\frac{\sigma}{\varkappa}}\right]}\frac{e^{\sigma t}d\sigma}{\sigma};\ c>0. \qquad (III.1.21)$$

. Integrand decreases with $|\sigma|\to\infty$. This makes it possible by usual reception/procedure to pass to integration for direct/straight, parallel imaginary axis, but that which lie to the left of it, after adding contributions from the poles of integrand, which lie between straight lines. ; therefore $\pi^2\varkappa L^{-2}<-c_1<4\pi^2\varkappa L^{-2}$, we have

$$P_1(x,\ t)=p^0\left(1-\frac{x}{L}\right)-\frac{2p^0}{\pi}\sin\left[\pi\left(1-\frac{x}{L}\right)\right]\exp\left(-\frac{\pi^2\varkappa t}{L^2}\right)+$$
$$+0\left[\exp\left(-\frac{(4-\varepsilon)\,\pi^2\varkappa t}{L^2}\right)\right]. \qquad (III.1.22)$$

The first term of expression (III.1.22) is the steady-state solution, which corresponds the assigned boundary conditions; the second term expresses basic, with long times, part of the correction to this steady-state solution; finally, the last/latter member is low even in comparison with the first correction term. Thus, approach/approximation to steady state occurs exponentially, whereupon characteristic time of output/yield to steady state - order

$$\tau = L^2 \varkappa^{-1} \ ^{-2}. \quad \text{(III.1.23)}$$

Let us estimate this time for the systems of different size/dimensions. In this case, for an $\varkappa$ let us accept the characteristic value of $\varkappa = 10^4$ cm²/s. As a result we have: with L = 1 m (transient process in one piece-block of rock/species) τ = 0.1 s; with L = 300 m (order of the distance between holes) $\tau = 10^4$ s $\approx$ 3 h; L = 10 km (order of the size/dimensions of deposit) $\tau = 10^7$ s = 100 days; L = 100 km (order of the size/dimensions of large water-pressure system) $\tau = 10^9$ s = 30 years. In practical problems frequently it is necessary to examine unsteady processes in the complex systems in which enter the cell/elements with different their own at times. Estimating set-up time (steady flow) for each cell/element according to its size/dimensions, we is simplified problem, after separate/liberating those cell/elements motion in which already it is possible to consider stationary, and those in which unsteady process is located in the initial stage.

Page 32.

§2. Axisymmetric problems and the problems of the interference of holes during the unsteady filtration.

1. Let us examine now one-dimensional axisymmetric (flat/plane-radial) motion during elastic mode/conditions. The pressure distribution is defined in this case as solution to the equation of thermal conductivity in polar coordinates (r, θ):

$$\frac{\partial p}{\partial t}=\varkappa\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\quad 0<\rho\leqslant r\leqslant R\leqslant\infty, \qquad \text{(III.2.1)}$$

satisfying the initial condition

$$p(r,\ 0)=f(r) \qquad \text{(III.2.2)}$$

and boundary conditions with $r = \rho$ and $r = R$.

As before basic interest they represent the solutions, which correspond the stationary initial distribution of pressure $f(r) = C_1 \ln r + C_2$. On the strength of the linearity of the equation (III.2.1) of the deviation of distribution, the pressures from the stationary also satisfy to equation (III.2.1), but already with zero initial condition. Therefore let us accept further $f(r) \equiv 0$, understanding by p the deviation of pressure from stationary distribution.

Transfer/converting in the equation (III.2.1) to Laplace transforms, we obtain equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dP(r,\sigma)}{dr}\right)=\frac{\sigma}{\varkappa}P(r,\sigma), \qquad \text{(III.2.3)}$$

general solution of which it takes the form:

$$P(r, \sigma) = C_1 I_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right) + C_2 K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right), \qquad \text{(III.2.4)}$$

where I_0 and K_0 are the modified Bessel functions of the apparent/imaginary zero-order argument. Assigning two boundary conditions, we can determine the constants C_0, C_2, and together with them and solution.

2. In appendices special importance has the problem in which on hole is assigned constant pressure, but constant output. The solution of this problem is utilized in the moswidely accepted methods of parameter determination of layer from the observations of unsteady inflow to hole. Let us rely thus

$$q = -\frac{2\pi k}{\mu}\left(r\frac{\partial p}{\partial r}\right)_{r=\rho} = \text{const};\quad \left.\frac{\partial p}{\partial r}\right|_{r=\rho} = \frac{p^*}{\rho};$$
$$p^* = -\frac{q\mu}{2\pi k};\quad p(R,\ t) = 0, \qquad \text{(III.2.5)}$$

where p* - the constant of dimensionality pressure.

Page 33.

Satisfying boundary conditions, we will obtain for the image of the distribution of pressure

$$P(\sigma) = \frac{p^*\sqrt{\varkappa}}{\rho\sigma^{3/2}}\,\frac{K_0\left(R\sqrt{\frac{\sigma}{\varkappa}}\right)I_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right) - K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right)I_0\left(R\sqrt{\frac{\sigma}{\varkappa}}\right)}{I_1\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)K_0\left(R\sqrt{\frac{\sigma}{\varkappa}}\right) + I_0\left(R\sqrt{\frac{\sigma}{\varkappa}}\right)K_1\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}. \qquad \text{(III.2.6)}$$

Let us examine "intermediate asymptotic behavior"

$$\frac{\rho}{\sqrt{\varkappa t}} \ll 1 \ll \frac{R}{\sqrt{\varkappa t}},$$

that it makes it possible to simplify expression (III.2.6), set/assuming

$$\rho\sqrt{\frac{\sigma}{\varkappa}} \ll 1 \ll R\sqrt{\frac{\sigma}{\varkappa}}.$$

At great significance of argument, the modified Bessel functions have asymptotic expressions [26, 129]

$$I_0(z) \approx \frac{1}{\sqrt{2\pi z}} e^z; \quad K_0(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \qquad \text{(III.2.7)}$$

in this case the boundary condition cn cuter duct turns out to be unessential, and we have solution for the unlimited layer:

$$P(r, \sigma) = -\frac{p^* \sqrt{\varkappa}}{\rho \sigma^{3/2}} \frac{K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right)}{K_1\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}. \qquad \text{(III.2.8)}$$

. is simplified now expression (III.2.8), utilizing an inequality of $\rho \ll \sqrt{\varkappa t}$ and asymptotic formulas for K_1 (z) and K_0 (z) with $z \to 0$:

$$K_0(z) = -\left(C + \ln\frac{z}{2}\right) \qquad \text{(III.2.9)}$$

(C = 0.7772 - is Euler's constant).

Substituting (III.2.9) in (III.2.8), we obtain

$$P(r,\sigma)=-\frac{p^*}{\sigma}K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right), \qquad (\mathrm{III}.2.10)$$

specifically, for a pressure in hole

$$P(\rho,\sigma)=\frac{p^*}{\sigma}\left(C+\ln\rho\sqrt{\frac{\sigma}{\varkappa}}\right). \qquad (\mathrm{III}.2.11)$$

Let us note the important fact: in relationship/ratios (III.2.10) and (III.2.11) does not enter a radius of hole ρ. This means that in the field of the applicability of the condition of $\rho^2/\varkappa t \ll 1$ pressure distribution does not depend on a radius of hole.

Page 34.

Utilizing table of the Laplace transforms (for example, see [26, formula 5.16 (35)], we have

$$K_0(\sqrt{\alpha\sigma}) \rightleftarrows \frac{1}{2t}\exp\left(-\frac{\alpha}{4t}\right). \qquad \text{(III.2.12)}$$

. Hence, by utilizing communication/connection between the Laplace transforms of Laplace function and its derivative] the formula (C.3) of application/appendix], we will obtain

$$\frac{1}{\sigma}K_0(\sqrt{\alpha\sigma}) \rightleftarrows \int_0^t \frac{1}{2\tau}\exp\left(-\frac{\alpha}{4\tau}\right)d\tau = \frac{1}{2}\int_{\alpha/4t}^{\infty} e^{-s}\frac{ds}{s} = \\ = -\frac{1}{2}\,\mathrm{Ei}\left(-\frac{\alpha}{4\tau}\right). \qquad \text{(III.2.13)}$$

. Set/assuming now in accordance with (III.2.10)

$\alpha = r^2/\varkappa$, we get

$$p(r,\ t) = \frac{1}{2} p^* \operatorname{Ei}\left(-\frac{r^2}{4\varkappa t}\right) = \frac{q\mu}{4\pi k} \operatorname{Ei}\left(-\frac{r^2}{4\varkappa t}\right). \quad \text{(III.2.14)}$$

Expression for a pressure in hole p (ρ, t) can be directly obtained from expression (III.2.11), on the tables of the Laplace transform or utilizing a known asymptotic behavior of exponential integral

$$\operatorname{Ei}(-x) = C + \ln x \ldots \quad \text{(III.2.15)}$$

In this case, we will obtain

$$p(\rho,\ t) = +\frac{q\mu}{4\pi k}\left(\ln \frac{r^2}{4\varkappa t} + C \ldots\right) \approx -\frac{q\mu}{4\pi k} \ln \frac{2{,}25\varkappa t}{r^2}. \quad \text{(III.2.16)}$$

The last/latter expression widely is utilized with the interpretation of the results of the investigation of holes (see §III.4).

3. Let us examine now the simplest problems of the interference of holes during unsteady motion. Interest in these problems is connected with that fact that on any deposit is a large number of holes, united into more or less correctly arranged/located groups (batteries), whereupon the mode/conditions of all holes in battery usually are approximately identical. during the calculation it is convenient to replace battery of uniform holes by drainage gallery either the amalgamated hole: the group of discrete flows (sources) to replace battery of uniform holes by drainage gallery or the amalgamated hole: the group of discrete flows (sources) is replaced by one distributed.

Let there be the infinite chain/network of the holes, arrange/located at a distance 2a from each other along the straight line which we accept for X-axis. Let us assume that at first the motions was not, but with t = 0 of all holes, begins selection with the identical output q. Let us examine pressure change in point with coordinates x, y, where for a certainty it is considered, that the $|x| < a, y > 0$ and that one of the holes is arranged in the

beginning of coordinates. Considering that a radius of hole is small in comparison with the distance between adjacent holes 2a, we will use the mentioned in the preceding/previous point/item expression for a pressure, obtained from self-similar solution.

Page 35.

We have

$$p = \frac{1}{2} p^* \sum_{n=-\infty}^{\infty} \mathrm{Ei}\left[-\frac{(x-2na)^2 + y^2}{4\varkappa t} \right]. \qquad (III.2.17)$$

. Practical interest usually they represent two question: the rate of a decompression in the holes of chain/network and the rate of a decompression at large removal/distance from chain/network. Let us rely therefore first y >> a. Let us compute pressure at the points, which lie on the axis y. We have

$$p(0, y) = \frac{1}{2} p^* \sum_{n=-\infty}^{\infty} \mathrm{Ei}\left(-\frac{4n^2a^2 + y^2}{4\varkappa t}\right) =$$

$$= \frac{1}{2} p^* \mathrm{Ei}\left(-\frac{y^2}{4\varkappa t}\right) - p^* \int_1^{\infty} \sum_{n=1}^{\infty} \exp\left(-\frac{un^2a^2}{\varkappa t}\right) \exp\left(-\frac{uy^2}{4\varkappa t}\right) \frac{du}{u}.$$

(III.2.18)

The last/latter expression can be converted, by utilizing an identity [41, No 552, 6]:

$$\sum_{k=1}^{\infty} \exp(-k^2x^2) = -\frac{1}{2} + \frac{\sqrt{\pi}}{x}\left[\frac{1}{2} + \sum_{k=1}^{\infty} \exp\left(-\frac{k^2\pi^2}{x^2}\right)\right].$$

(III.2.19)

We have

$$p(0,\ y) = -\frac{1}{2} p^* \int\limits_{\frac{a^2}{\pi\varkappa t}}^{\infty} \exp\left(-\frac{\pi y^2 v}{4a^2}\right) \frac{dv}{v^{3/2}} -$$

$$- p^* \int\limits_{\frac{a^2}{\pi\varkappa t}}^{\infty} \exp\left(-\frac{\pi y^2 v}{4a^2}\right) \sum_{k=1}^{\infty} \exp\left(-\frac{\pi k^2}{v}\right) \frac{dv}{v^{3/2}}. \qquad \text{(III.2.20)}$$

In the last/latter integral, on the strength of its rapid convergence on lower limit, possible the integration of news from zero. After this the result of term-by-term integration of a series can be found sufficiently simply. We have (for example it is direct on the tables of the Laplace transform)

$$\int\limits_{0}^{\infty} \exp\left(-\frac{\pi y^2 v}{4a^2} - \frac{\pi k^2}{v}\right) \frac{dv}{v^{3/2}} = \frac{1}{k} \exp\left(-\frac{\pi k y}{a}\right). \qquad \text{(III.2.21)}$$

. Page 36. As a result the second integral in equation (III.2.20) is represented in the form of integral of geometric progression. In the first integral it is convenient to isolate not decreasing with t → ∞ terms. After the indicated transformations we have

$$p(0,\ y)=p^{*}\left[-\frac{\sqrt{\pi\varkappa t}}{a}+\frac{\pi y}{2a}+\ln\left(1-e^{-\pi y/a}\right)+o(1)\right]. \tag{III.2.22}$$

. Recall that the formula (III.2.22) describes pressure distribution along the line, passing through axis of one of the holes and the perpendicular axis of battery, during sufficiently long times of $4\varkappa t \gg y^2$, $4\varkappa t \gg a^2$.

We see that far from hole (with y >> a) the last/latter term of expression (III.2.22) is unessential, the first two member of this to expressions if we consider the condition of $y^2 \ll 4\varkappa t$, coincide with expression for a pressure drop during the launching/starting of gallery with output q/4a taking into account the unit of area of gallery. virtually this expression it is possible to use already by beginning with Y = a.

Is of interest the explanation how pressure is changed along the lines, parallel the axes of gallery. It is directly obvious that

$$p(0,\ y) \leqslant p(x,\ y) \leqslant p(a,\ y), \qquad \text{(III.2.23)}$$

whence it is easy to obtain the value of fluctuation of pressure along line y = const. We have

$$p(a,\ y) = \frac{1}{2} p^{*} \left[\sum_{k=-\infty}^{\infty} \mathrm{Ei}\left(-\frac{y^2 + k^2 a^2}{4\varkappa t}\right) - \sum_{k=-\infty}^{\infty} \mathrm{Ei}\left(-\frac{y^2 + 4k^2 a^2}{4\varkappa t}\right) \right].$$

If we designate is temporary through the p_a' pressure distribution, determined by formula (III.2.22), then

$$p(a,\ y) = p'_{\frac{a}{2}} - p'_a = p^* \left[-\frac{\sqrt{\pi \varkappa t}}{a} + \right.$$

$$\left. + \frac{\pi y}{2a} + \ln \frac{1 - \exp\left(-\frac{2\pi y}{a}\right)}{1 - \exp\left(-\frac{\pi y}{a}\right)} + o(1) \right]. \qquad \text{(III.2.24)}$$

The comparison of expressions (III.2.22) and (III.2.24) gives

$$p(a,\ y)-p(0,\ y)=p^{*}\left[\ln\frac{1-e^{\frac{-2\pi y}{a}}}{1-e^{-\frac{\pi y}{a}}}+o(1)\right]. \qquad (III.2.25)$$

. Page 37.

Thus, with removal/distance from the battery of holes the difference in pressures between isclated pcints direct/straight y = const rapidly disappears and already with y = a it can be disregarded.

The obtained results show that outside the band of $|y|\leqslant a$ mction can be with high degree of accuracy considered cne-dimensional; on the contrary, within this band is essential the ncn-univariate character of motion, connected with the presence of pcint flows (holes) instead of those which were distributed.

Let us compute, by utilizing a formula (III.2.22), pressure in hole, after placing Y = ρ << a - a radius of hole). We have

$$p(0, \rho) = -p^*\left(\frac{\sqrt{\pi\varkappa t}}{a} - \ln\frac{\pi\rho}{a} + \ldots\right). \qquad (III.2.26)$$

Expression (III.2.26) shows that the pressure in hole differs from pressure on gallery with that output on the unit of the sectional area of layer only by constant component p* ln ($\pi\rho$/a). This fact makes it possible to conduct the calculation of the batteries of holes just as calculation of galleries, adding to the pressure differential value p* ln (πr/a).

The corresponding method (the "method of filtration resistance") is developed by Yu. P. Borisov [33] initially for steady motion; his application/use to unsteady processes given in work [113]. Method is reduced to the fact that the resistance to inflow to holes is divide/marked off into two been connected in series: the external

resistance, which corresponds motion outside gallery, and internal, that determines a pressure difference between the hole and the fictitious gallery. In connection with unsteady motion especially important simplification is achieved because of the fact that the transiency one should consider only for an external pressure: additional resistance upon transition from hole to gallery can be considered constant.

§3. Some special problems of elastic mode/conditions.

Let us examine now several more complex problems, which have vital importance for application/appendices.

1. Let us assume that the flat/plane layer consists of two fields with different properties (permeability, porosity, etc.), divided by rectilinear boundary (Fig. III.1). Let at torque/moment t = 0 initially steady state be agitated as a result of the launching/starting of hole at point (a, 0) with constant flow rate/consumption of q. Then pressure distribution in each of the fields is described by equations

Page 38.

$$\left.\begin{aligned}\frac{1}{\varkappa_2}\,\frac{\partial p}{\partial t}&=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}; \quad (x<0);\\ \frac{1}{\varkappa_1}\,\frac{\partial p}{\partial t}&=\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}-\frac{\mu}{k}\,q\delta(y)\,\delta(x-a); \quad (x>0).\end{aligned}\right\} \qquad \text{(III.3.1)}$$

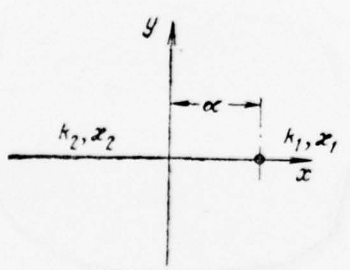


Fig. III.1.

With boundary of x = 0, are satisfied the continuity conditions of pressures and flows:

$$p(x-0,\ y)=p(x+0,\ y);\quad k_1\frac{\partial p(x+0,\ y)}{\partial x}=k_2\cdot\frac{\partial p(x-0,\ y)}{\partial x}. \tag{III.3.2}$$

In the second equation (III.3.1) into right side is directly introduced the point source with an intensity of q. By δ, as usual, is designated the delta function, determined by conditions (for example, see [106])

$$\int_{-\infty}^{\infty}\delta(\xi)f(\xi)\,d\xi=f(0);\quad \delta(\xi)=0;\quad (\xi\neq 0). \tag{III.3.3}$$

After using to equations (III.3.1) the cosine-conversion of Fourier and Laplace transforms, we will obtain

$$\left.\begin{aligned}&\frac{d^2P}{dx^2}-\left(\omega^2+\frac{\sigma}{\varkappa_2}\right)P=0; \quad (x<0);\\&\frac{d^2P}{dx^2}-\left(\omega^2+\frac{\sigma}{\varkappa_1}\right)P=\frac{\mu q}{2k_1\sigma}\,\delta(x-a),\end{aligned}\right\} \tag{III.3.4}$$

where

$$P(x,\ \omega,\ \sigma)=\int_0^\infty \cos\omega y\,dy\int_0^\infty e^{-\sigma t}\,p(x,\ y,\ t)\,dt. \tag{III.3.5}$$

The presence in the right side of the second order equation (III.3.4) of the value of $\left(\frac{\mu q}{2k_1\sigma}\right)\delta(x-a)$ means that P satisfies homogeneous equation with $x\neq a$, and with x = a the derivative dP/dx it undergoes jump by the value of $\frac{\mu q}{2k_1\sigma}$, whereas value itself P is continuous.

taking into account this, is easy to extract the solution to equations (III.3.4), satisfy the conditions of seam in x = 0 and vanishing in $x \to \pm\infty$. This solution takes the form:

$$\left.\begin{aligned} P(x,\ \omega,\ \sigma) &= C\left[\operatorname{ch} s_1 x + \frac{k_2}{k_1}\frac{s_2}{s_1}\operatorname{sh} s_1 x\right]; \quad (0<x<a);\\ P(x,\ \omega,\ \sigma) &= A e^{-s_1 x}; \quad (a<x<\infty);\\ P(x,\ \omega,\ \sigma) &= C e^{s_2 x}; \quad (x<0);\\ s_1 &= \sqrt{\omega^2+\frac{\sigma}{\varkappa_1}}; \quad s_2=\sqrt{\omega^2+\frac{\sigma}{\varkappa_2}}, \end{aligned}\right\} \quad \text{(III.3.6)}$$

where

$$\left.\begin{aligned} C &= -\frac{\mu q}{2k_1\sigma}\,\frac{e^{-s_1 a}}{s_1+\dfrac{k_2 s_2}{k_1}};\\ A &= -\frac{\mu q}{2k_1\sigma}\,\frac{k_1\operatorname{ch} s_1 a+\dfrac{k_2 s_2}{s_1}\operatorname{sh} s_1 a}{k_1 s_1+k_2 s_2}. \end{aligned}\right\} \quad \text{(III.3.7)}$$

. Page 39.

We examine the now obtained solution. Let us determine first pressure at point $x = a + \rho$, $y = 0$ ($\rho \ll a$).

We have

$$P(a+\rho) = -\frac{\mu q}{4k_1\sigma}\frac{e^{-s_1\rho}}{s_1}\left[1+\frac{k_1s_1-k_2s_2}{k_1s_1+k_2s_2}e^{-2as_1}\right]. \qquad \text{(III.3.8)}$$

Let the $\sigma a^2/\varkappa_1 \gg 1$ (i.e. be examined short times; $t \ll a^2/\varkappa_1$). In this case

$$P(a+\rho) \approx -\frac{\mu q}{4k_1\sigma}\frac{e^{-\rho s_1}}{s_1}. \qquad \text{(III.3.9)}$$

In order to calculate the Laplace transform of pressure, it is necessary to fulfill the reverse/inverse cosine-conversion:

$$\bar{p}(x,\ y,\ \sigma) = \frac{2}{\pi}\int_0^\infty P\cos\omega y\,d\omega\,, \qquad \text{(III.3.10)}$$

whence

$$\bar{p}(a+\rho, 0, \sigma)=\frac{2}{\pi}\int_0^\infty P(a+\rho)\,d\omega. \qquad \text{(III.3.11)}$$

Substituting (III.3.9) in (III.3.11), we have

$$\bar{p}(a+\rho, 0, \sigma)=-\frac{\mu q}{2\pi k_1\sigma}\int_0^\infty \frac{e^{-\rho\sqrt{\omega^2+\frac{\sigma}{\varkappa_1}}}}{\sqrt{\omega^2+\frac{\sigma}{\varkappa_1}}}\,d\omega=$$

$$=-\frac{\mu q e^{-\rho\sqrt{\frac{\sigma}{\varkappa_1}}}}{2\pi k_1\sigma}\int_0^\infty \frac{e^{-u\rho\sqrt{\frac{\sigma}{\varkappa_1}}}}{\sqrt{u(u+2)}}\,du=-\frac{\mu q}{2\pi k_1\sigma}K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\rho\right). \qquad \text{(III.3.12)}$$

We utilize a known relationship/ratic

$$K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\,\rho\right) \leftarrow\div\rightarrow \frac{1}{2t}\exp\left(-\frac{\rho^2}{4\varkappa_1 t}\right).$$

. Then

$$\frac{1}{\sigma}K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\,\rho\right) \leftarrow\div\rightarrow \frac{1}{2}\int_0^t \frac{1}{t}\exp\left(-\frac{\rho^2}{4\varkappa_1 t}\right)dt = -\frac{1}{2}\,\mathrm{Ei}\left(-\frac{\rho^2}{4\varkappa_1 t}\right). \tag{III.3.13}$$

.

Table III-1.

k_1/k_2	Δ_0	k_1/k_2	Δ_0
0,01	−0,022	1	0
0,1	−0,195	2	0,267
0,2	−0,120	3	0,534
0,333	−0,123	5	1,030
0,5	−0,106	10	2,080
		100	11,400

The obtained result, of course, is completely evident; it means that for sufficiently short times of $(t^2 \ll a^2/\varkappa_1)$ the effect of the second zone can be disregarded, and pressure is distributed just as in uniform layer.

Let there be now the reverse/inverse inequality: $t^2 \gg a^2/\varkappa_1$, so that $\sigma \ll \varkappa_1/a^2$. In this case, the contribution of first term (III.3.8) will not change; the contribution of the second term no longer will be negligible. Let us present it in the form:

$$-\frac{\mu q}{4k_1\sigma}\,\frac{\exp[-(2a+\rho)s_1]}{s_1}\left(\frac{k_1-k_2}{k_1+k_2}+\frac{2k_1k_2(s_1-s_2)}{(k_1+k_2)(k_1s_1+k_2s_2)}\right). \quad \text{(III.3.14)}$$

The first term (III.3.14) similar to expression (III.3.9) answers source flow, located at point (-a, 0) and released into torque/moment t = 0 with output $q\frac{k_1-k_2}{k_1+k_2}$; its contribution can be easily calculated. The second term (III.3.14) gives after inversion

$$\Delta(a+\rho, \sigma) = -\frac{\mu q}{\pi(k_1+k_2)}\frac{k_2}{\sigma}\int_0^\infty \frac{\exp[-(2a+\rho)s_1]}{s_1}\frac{s_1-s_2}{k_1 s_1+k_2 s_2}\,d\omega. \quad \text{(III.3.15)}$$

Integral (III.3.15) takes with $\sigma = 0$ finite value $1/k_1\Delta_0$. This means that during long times pressure distribution takes the very simple form, whereupon the omitted terms vanish with $t \to \infty$:

$$p(x, y, t) = \frac{\mu q}{4\pi k_1}\left[\mathrm{Ei}\left(-\frac{\rho^2}{4\varkappa_1 t}\right) + \frac{k_1-k_2}{k_1+k_2}\mathrm{Ei}\left(-\frac{(2a+\rho)^2}{4\varkappa_1 t}\right)\right] - \\ -\frac{\mu q k_2}{\pi(k_1+k_2)k_1}\Delta_0(\beta, k_2/k_1); \quad (\beta^2 = \varkappa_1/\varkappa_2). \quad \text{(III.3.16)}$$

Table III.1 gives the values of integral Δ_0 for the case $\beta^2 = k_1/k_2$.

P. Ya. Polubarinova-Kochina [94] examined analogous stationary

problem and showed that pressure distribution in that part of the layer, where is located source, coincides with pressure distribution in the homogeneous medium during the action of two sources with an intensity of q and λq (where $\lambda = k_1 - k_2/k_1 + k_2$), arrange/located symmetrically relative to the boundary of section.

Page 41.

As we saw above, the same it occurs also for motion during elastic mode/conditions. In this case, it is necessary to keep in mind that the appearance of an additive term Δ_0 does not contradict the aforesaid, since during steady motion the pressure is determined with an accuracy to constant; during the unsteady motion of this arbitrariness already no, since by natural reference point serves the pressure in unperturbed layer. The examined problem studied for the first time V. A. Maximov [75, 3].

The given simplified form of solution is convenient during the analysis of solution for the section of layer near hole. for the moved away sections of layer, the solution can be presented in another way. Is feasible this for the points, arrange/located on the

axis y, counting y >> a. With this P (0, ω, σ) = C and

$$\bar{p}(0, y, \sigma) = -\frac{\mu q}{\pi k_1 \sigma} \int_0^\infty \frac{e^{-as_1} \cos \omega y \, d\omega}{s_1 + (k_2/k_1) s_2}. \qquad \text{(III.3.17)}$$

With y >> a $a^2\sigma/\varkappa_1 \ll 1$ the dominant term of expression (III.3.17) takes the form:

$$\bar{p}(0, y, \sigma) = -\frac{\mu q}{\pi k_1 \sigma} \int_0^\infty \frac{\cos \omega y \, d\omega}{s_1 + k_2/k_1 s_2} =$$

$$= -\frac{\mu q}{\pi\sigma (k_1 + k_2)} K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\, y\right) - \frac{\mu q k_2}{k_1 \pi \sigma (k_1 + k_2)} \int_0^\infty \frac{\cos \omega y \, (s_1 - s_2) \, d\omega}{s_1 (s_1 + k_2/k_1 s_2)} =$$

$$= -\frac{\mu q}{\pi\sigma (k_1 + k_2)} K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\, y\right) - \frac{\mu q k_2 (\varkappa_2 - \varkappa_1)}{\pi \sigma k_1 (k_1 + k_2) \varkappa_2} F(\eta, \beta, k_2/k_1);$$

$$F(\eta, \beta, k_2/k_1) = \int_0^\infty \frac{\cos \eta\nu \, d\nu}{\sqrt{1+\nu^2}\,\left(\sqrt{1+\nu^2} + \sqrt{\beta^2+\nu^2}\right)\left(\sqrt{1+\nu^2} + k_2/k_1 \sqrt{\beta^2+\nu^2}\right)};$$

$$\left(\omega = \nu \sqrt{\sigma/\varkappa_1}; \quad \beta^2 = \varkappa_1/\varkappa_2; \quad \eta = y \sqrt{\sigma/\varkappa_1}\right). \qquad \text{(III.3.18)}$$

The obtained asymptotic behavior makes simple sense. If the point at which is examined the solution, is moved away from the origin of coordinates up to the distance, greater in comparison with a, then a precise position of hole no longer it is important. Specifically, it is possible in the first approximation, to consider hole arrange/located directly on the boundary of layers. In the problem whose form has thus been changed already is absent significant dimension, and it has the self-similar solution of the form:

$$p(x, y, t) = \frac{\mu q}{k_1} f(\xi, \Theta), \qquad \text{(III.3.19)}$$

where $\xi = \frac{r}{2\sqrt{\varkappa_1 t}}$; r and θ - polar coordinates.

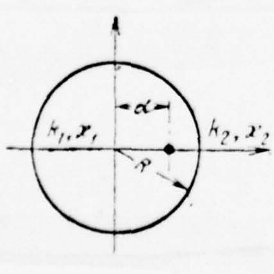


Fig. III.2.

The given expression (III.3.18) answers value $\theta = \pi/2$.

It is possible to refine the examined asymptotic behavior, by taking into account the following terms of expansion.

In order to understand the sense of the consequents of expansion, let us note that, "shift/shearing" source in the beginning of coordinates (i.e. set/assuming a = 0), we insert the error which can be compensated for, after adding "pair" from flow in the beginning of coordinates and source at point (a, 0). The action of this pair it is possible to approximately replace with the action of "dipole" by intensity qa, arrange/located in the beginning of coordinates. Correspond it solution also is self-similar. Similarly can be produced the account of higher corrections. As a result the solution for large r will turn out to be that which was presented in the form of the imposition of the self-similar solutions of a comparatively simple form.

2. Let us examine now the analogous problem of the effect of the heterogeneity of layer, by considering that the heterogeneity has circular form (Fig. III.2). Have

$$\frac{1}{\varkappa_1}\frac{\partial p}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial\Theta^2}-\frac{\mu q}{k_1 r}\delta(r-a)\delta(\Theta)\quad(r<R);$$
$$\frac{1}{\varkappa_2}\frac{\partial p}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial\Theta^2}\quad(r>R),\qquad\text{(III.3.20)}$$

a with boundary of r = R are fulfilled the condition of the seam:

$$p(R-0,\Theta,t)=p(R+0,\Theta,t);$$
$$k_1\frac{\partial p(R-0,\Theta,t)}{\partial r}=k_2\frac{\partial p(R+0,\Theta,t)}{\partial r}.\qquad\text{(III.3.21)}$$

After presenting solution in Fourier series, we will obtain for coefficients

$$P_n(r, \sigma) = \frac{2}{\pi} \int_0^{\pi} \int_0^{\infty} e^{-\sigma t} p(r, \Theta, t) \cos n\Theta \, dt \, d\Theta \qquad \text{(III.3.22)}$$

expression

$$P_n = C I_n \left(\sqrt{\sigma/\varkappa_1}\, r\right) \qquad (r < a); \qquad \text{(III.3.23)}$$

$$P_n = B I_n \left(\sqrt{\sigma/\varkappa_1}\, r\right) + D K_n \left(\sqrt{\sigma/\varkappa_1}\, r\right) \qquad (a < r < R);$$

$$P_n = A K_n \left(\sqrt{\sigma/\varkappa_1}\, r\right) \qquad (r > R).$$

Here

$$B = \frac{\mu q}{\pi k_1 \sigma} I_n(\alpha) \times$$

$$\times \frac{K_n(\rho_0)\,[K_{n+1}(\beta\rho_0) + K_{n-1}(\beta\rho_0)] - \beta K_{n+1}(\rho_0)\, K_n(\beta\rho_0) - \beta K_n(\rho_0)\, K_{n-1}(\beta\rho_0)}{I_n(\rho_0)\,[K_{n+1}(\beta\rho_0) + K_{n-1}(\beta\rho_0)] + \beta K_n(\beta\rho_0)\,[I_{n+1}(\rho_0) + I_{n-1}(\rho_0)]};$$

$$\left.\begin{aligned}
&C=B-\frac{\mu q}{\pi k_1\sigma}K_n(\alpha);\quad D=-\frac{\mu q}{\pi k_1\sigma}I_n'(\alpha);\\
&A=B\frac{I_n(\rho_0)}{K_n(\beta\rho_0)}+D\frac{K_n(\rho_0)}{K_n(\beta\rho_0)};\\
&\alpha=\sqrt{\frac{\sigma}{\varkappa_1}}a;\quad \rho_0=\sqrt{\frac{\sigma}{\varkappa_1}}R;\quad \beta^2=\frac{\varkappa_2}{\varkappa_1};\quad \rho=\sqrt{\frac{\sigma}{\varkappa_1}}r.
\end{aligned}\right\} \qquad \text{(III.3.24)}$$

. Page 43.

Let us select point $r = a - \delta$, $0 < \delta \ll a$. We have

$$P_n(a-\delta)=BI_n(\rho)-\frac{\mu q}{\pi k_1\sigma}K_n(\alpha)I_n(\rho). \qquad \text{(III.3.25)}$$

The corresponding to the second member of expansion (III.3.25) Fourier series we sum. Then

$$\bar{p}(\rho,\Theta,\sigma)=\frac{1}{2}P_0+\sum_{n=1}^{\infty}P_n\cos n\Theta=$$
$$=-\frac{\mu q}{\pi k_1\sigma}\left(\frac{1}{2}K_0(\alpha)I_0(\rho)+\sum_{n=1}^{\infty}K_n(\alpha)I_n(\rho)\cos n\Theta\right)=-\frac{\mu q}{2\pi k_1\sigma}K_0(\omega);$$
$$\omega=\sqrt{\rho^2+\alpha^2-2\alpha\rho\cos\Theta} \qquad \text{(III.3.26)}$$

(see [26, formula 7, 15, 35]), which answers source flow in uniform layer. If α and ρ_0 are great (σ are great, i.e., times shafts), then as it is easy to verify that the contribution of the first term of formula (III.3.25) is exponentially small, so that solution coincides with solution for a uniform layer. This occurs, obviously, until

compressive disturbance achieves the circular boundary of r = R.

Let us examine in more detail the opposite case when α, ρ_0 and $\beta\rho_0$ are small. then expression (III.3.24) for B can be simplified, by utilizing known formulas for the low values of the argument:

$$I_n(\alpha)\approx\frac{\alpha^n}{2^n n!};\qquad K_n(\alpha)\approx\frac{(n-1)!\,2^{n-1}}{\alpha^n}\quad (n>0).\quad \text{(III.3.27)}$$

As a result with $n\neq0$ in (III.3.24) it is possible to disregard the terms, which contain K_{n-1} and I_{n+1}, and to obtain after some simplifications

$$B_n=\frac{\mu q}{2\pi k_1\sigma}\,\frac{k_2-k_1}{k_2+k_1}\,2^n(n-1)!\left(\frac{\alpha}{\rho_0^2}\right)^n.\qquad \text{(III.3.28)}$$

With n = 0, we have

$$B_0 = \frac{\mu q}{\pi k_1 \sigma} \left[\left(\frac{k_1}{k_2} - 1 \right) \left(C + \ln \frac{\rho_0}{2} \right) + \frac{k_1}{k_2} \ln \beta \right]. \quad \text{(III.3.29)}$$

. Page 44.

The contribution of expression (III.3.28) to solution is equal to

$$- \frac{k_1 - k_2}{k_1 + k_2} \frac{\mu q}{2\pi k_1 \sigma} \sum_{n=1}^{\infty} \left(\frac{\alpha \rho}{\rho_0^2} \right)^n \frac{\cos n\Theta}{n} =$$

$$= \frac{k_1 - k_2}{k_1 + k_2} \frac{\mu q}{4\pi k_1 \sigma} \ln \left(1 - \frac{2\alpha\rho}{\rho_0^2} \cos \Theta + \frac{\rho^2 \alpha^2}{\rho_0^4} \right) \quad \text{(III.3.30)}$$

(see [40, formula I.448.2]).

Thus, the corresponding great significance of time image of pressure distribution takes the form:

$$\bar{p}(r, \Theta, \sigma) = \frac{\mu q}{2\pi k_1 \sigma}\left[\ln \frac{\sqrt{r^2 + a^2 - 2ar\cos\Theta}}{R} + \right.$$
$$\left. + \frac{k_1 - k_2}{k_1 + k_2} \ln \frac{\sqrt{r^2 a^2 + R^4 - 2arR^2\cos\Theta}}{R^2}\right] +$$
$$+ \frac{\mu q}{2k_2 \pi \sigma}\left[\ln \frac{R}{2}\sqrt{\frac{\sigma}{\varkappa_2}} + C + \ln \frac{\varkappa_2}{\varkappa_1}\right]. \qquad \text{(III.3.31)}$$

In formula (III.3.31) the expression in brackets with an accuracy to constant coincides with expression for pressure distribution in the appropriate stationary problem; the term, which contains $\ln \frac{R}{2}\sqrt{\frac{\sigma}{\varkappa_2}}$, during inverse transformation gives logarithmic growth in pressure in time, so that formula (III.3.31) detects as a whole the following picture: pressure in internal circular zone is distributed stationary, i.e., a pressure difference between any two points of field has the same value, as during steady motion, but the middle pressure level slowly (logarithmic) changes with an increase of time. In this case, the pressure only in terms of

a constant value differs from pressure near the single hole, released with the constant selection q in uniform layer by permeability k_2 and by the piezoconductivity of $\varkappa_2$.

Let us examine now the inflow of liquid to vertical crack by the length 2L, which we will consider the surface of constant pressure. let us assume that the complete selection of the liquid through the crack retains known constant value. Let us accept, as before the initial pressure in layer for zero. In this case,, taking into account symmetry relative to x and y axes, we have

$$\frac{\partial p}{\partial t}=\varkappa\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)\qquad(0<x<\infty;\quad 0<y<\infty);\quad \text{(III.3.32)}$$

$$\frac{\partial p\,(0,\,y)}{\partial x}=0;\quad \frac{\partial p\,(x,\,0)}{\partial x}=0\qquad(0<x<L);$$

$$\frac{\partial p\,(x,\,0)}{\partial y}=0\quad(L<x<\infty);\qquad \int\limits_0^L\frac{\partial p\,(x,\,0)}{\partial y}\,dx=\frac{\mu q}{2k}.$$

• Page 45.

Let us designate

$$\frac{\partial p\,(x,\,0)}{\partial y}=\rho\,(x);\qquad \int\limits_{-L}^{L}\rho\,(x)\,dx=\frac{\mu q}{k}.\qquad \text{(III.3.33)}$$

By considering value ρ (x) assigned, it is possible, by applying to problem (III.3.32) the Laplace transform in t and the cosine-conversion of Fourier in x, to find for the converted according to Laplace pressure P (x, y, σ) expression

$$P(x, y, \sigma) = -\frac{2}{\pi}\int_0^\infty \cos\omega x \frac{e^{-\sqrt{\omega^2+\frac{\sigma}{\varkappa}}\,y}}{\sqrt{\omega^2+\frac{\sigma}{\varkappa}}}\int_0^L \bar{\rho}(\xi)\cos\omega\xi\,d\xi\,d\omega. \quad (III.3.34)$$

This expression is the general solution of problem (III.3.32), satisfying all supplementary conditions, except the condition of pressure constancy on cut $0 < x < L$, $y = 0$. This condition gives

$$P(x, 0, \sigma) = P_0(\sigma) = -\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\, d\omega}{\sqrt{\omega^2+\frac{\sigma}{\varkappa}}}\int_0^L \bar{\rho}(\xi)\cos\omega\xi\, d\xi. \qquad \text{(III.3.35)}$$

Changing in this expression the order of integration and calculating internal integral

$$\int_0^\infty \frac{\cos\omega x \cos\omega\xi\, d\omega}{\sqrt{\omega^2+\frac{\sigma}{\varkappa}}} = \frac{1}{2}\left\{K_0\left[(x+\xi)\sqrt{\frac{\sigma}{\varkappa}}\right] + K_0\left[(x-\xi)\sqrt{\frac{\sigma}{\varkappa}}\right]\right\},$$

let us give this equation to the form:

$$P_0(\sigma) = -\frac{1}{\pi}\int_{-L}^{L} \bar{\rho}(\xi, \sigma) K_0\left[(x-\xi)\sqrt{\frac{\sigma}{\varkappa}}\right] d\xi, \qquad |x|<L$$

(here is additionally taken into account the parity of value ρ on the ξ, the directly following from the symmetry of problem). It is

convenient to introduce here dimensionless variable

$u=x/L, \eta=\xi/L, \lambda=L\sqrt{\frac{\sigma}{\varkappa}}$. In this case, we have

$$P_0(\lambda)=-\frac{L}{\pi}\int_{-1}^{1}\bar{\rho}(\eta,\lambda)K_0[(u-\eta)\lambda]\,d\eta,\quad |u|<1. \qquad \text{(III.3.36)}$$

. Formula (III.3.36) is integral equation for determining the unknown function of $\bar{\rho}(\eta,\lambda)$. We limit ourselves to the search of its approximate solution for small λ.

Page 46.

utilizing an asymptotic expression for K_0 at the low values of argument, we have

$$P_0(\lambda)=\frac{L}{\pi}\int_{-1}^{1}\bar{\rho}(\eta,\lambda)\left[\ln\frac{|u-\eta|\lambda}{2}+C+\ldots\right]d\eta \qquad \text{(III.3.37)}$$

or, taking into account (III.3.33),

$$\frac{L}{\pi}\int_{-1}^{1}\bar{\rho}(\eta,\lambda)\ln|u-\eta|\,d\eta = P_0 - \frac{\mu q}{\pi k\sigma}\left(\ln\frac{\lambda}{2}+C\right). \quad \text{(III.3.38)}$$

Let us note that if instead of the problem (III.3.32) we examined the same problem for steady motion, then, acting in that manner, they would obtain the integral equation of the form:

$$\frac{L}{\pi}\int_{-1}^{1}\rho_0(\eta)\ln|u-\eta|\,d\eta = \text{const}; \qquad \int_{-1}^{1}\rho_0(\eta)\,d\eta = \frac{\mu q}{kL}. \quad \text{(III.3.39)}$$

In this case, the constant in equation (III.3.39) is not determined, since in the appropriate stationary problem the pressure is determined only with an accuracy to constant. The stationary purpose can be easily achieved, by considering it as boundary-value problem of function theory, and by applying the formula of Keldysh - Sedov ([63, chapter III, §3]).

The unknown solution takes the form:

$$\rho_0(\eta) = \frac{\mu q}{\pi k L \sqrt{1-\eta^2}}. \qquad \text{(III.3.40)}$$

Equations (III.3.38) and (III.3.39) are equivalent, so that function of $\frac{1}{\sigma}\rho_0(\eta)$ gives in the approach/approximation in question also the solution of the initial problem. in other words, with λ << 1, that answers long times of $(t \gg L^2/\varkappa)$, the distribution of rates of filtration according to the surface of crack it is possible to consider stationary. By substituting in (III.3.38) $\bar{\rho} = \frac{1}{\sigma}\rho_0(\eta)$, we will obtain

$$P_0(\sigma) = \frac{\mu}{\pi\sigma k}\left(\ln\frac{\lambda}{2} + C\right) + \frac{\mu q}{k\sigma\pi^2}\int_{-1}^{1}\frac{\ln|u-\eta|\,d\eta}{\sqrt{1-\eta^2}} = \frac{q\mu}{\pi k\sigma}\left(\ln\frac{\lambda}{4} + C\right). \qquad \text{(III.3.41)}$$

The formula (III.3.41) determines the asymptotic behavior of pressure in crack with long times. Comparing it with formula (III.2.27), we see that the pressure changes just as in hole with a radius of $r_* = \frac{1}{2}L$, i.e.

$$p_0(t) = \frac{q\mu}{2\pi k}\left[\ln\frac{L^2}{16\varkappa t} + C + o(1)\right]. \qquad \text{(III.3.42)}$$

. Page 47.

This fact makes it possible to determine the size/dimension of crack by the observations of unsteady inflow to it (see §4).

4. The given examples make it possible to make some conclusions of common character relative to unsteady flows with complex geometry. In the majority of cases, the problems in question are reduced to research on the behavior of system under the localized influence sufficient simple form (for example during an abrupt change of the output in one of the holes).

From the obtained thus solutions in wish it is possible, by using the principle of superposition, to find the result of more

complex effects. This concentration of changes in time and in space leads to the fact that the asymptotic properties of the corresponding solutions turn out to be very simple. In the majority of problems, succeed in isolating three basic fields. The first of them - the moved away from the place of disturbance/perturbation field which to this moment still retains its initial state and the conditions in which still not had time to influence the behavior of solution. The second field - directly adjacent to the place of disturbance/perturbation; motion here in known sense is close to stationary (for example if disturbance/perturbation entailed an abrupt change in the flow rate through the crack (p. 3), then velocity field near crack is close to the stationary field of velocities). finally, in the third, comparatively narrow and having the simple geometric form of transient region occurs strictly unsteady motion. This simple structure of solution (is inherent moreover not only to the problems of elastic mode/conditions, but also to nonlinear unsteady problems) makes it possible in a number of cases easy to establish/install the basic course of solution, utilizing known steady-state solutions for the "internal" part of the range of motion and simple (for example self-similar) unsteady solutions for an exterior.

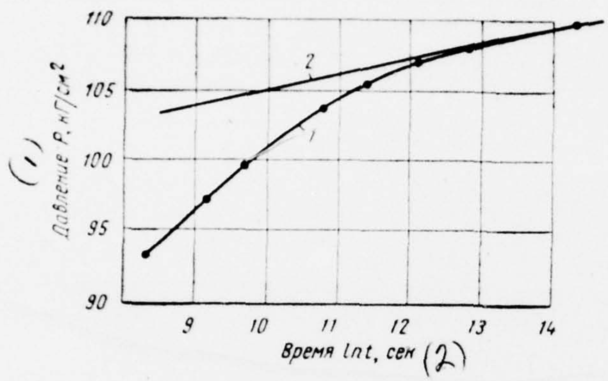
§4. Inverse problems of the linear theory of unsteady filtration.

In the questions of geology, especially geologies of oil, widely are utilized the solution to the inverse (stationary and unsteady) problems of filtration theory. The common/general/total principle of the study of layers during unsteady flow entails the fact that by changing the work of holes in layer artificially is created the nonstationary system of filtration and is measured pressure in time dependence in one or several holes. on the basis of the data on a change in the outputs of holes and side-looking pressure change in the fixed points of layer can be obtained the information about the parameters of layer - permeability, piezoconductivity, about the location of the boundaries of layer, etc.

1. The quite simple and most widely used method of designing of unsteady flow is cessation of one of the holes.

Fig. III.3.

Key: (1). Pressure P, kgf/cm². (2). Time Lnt, s.



The curve of the dependence of pressure on time with the aid of which is carried out the investigation of layer, can be removed either in the most stationary hole or in another - incperative hole (piezometric). In this case, it is assumed that during investigation the outputs of remaining holes are changed insignificantly. The typical form of the curve of pressure change in the stationary hole (recovery characteristic of pressure) is depicted on Fig. III.3 in coordinates p, ln t (is curve 1).

The solution of the problem of change to a constant value of pressure in infinite layer after a change of the output of hole was given in §2. From the viewpoint of the analysis of the recovery characteristics of pressure first of all, is of interest asymptotic section by the curve of p (ln t) with large t, described by formula (III.2.28). Pressure change in the hole, stopped after it worked in steady state with output q_0, it can be written in the form:

$$p(\rho, t) - p(\rho, 0) = \frac{q_0\mu}{4\pi kh}\left(\ln t + \ln \frac{4\varkappa}{\gamma\rho^2}\right). \qquad \text{(III.4.1)}$$

The formula (III.4.1) determines straight line in coordinates

p, ln t. During the plotting of curves of the pressure recovery in the stopped hole this straight portion frequently it is establish/installed within brief time (is curve 2). Let the equation of asymptote is p = A ln t + B. Then comparison with formula (III.4.1) shows that $A = \frac{q_0\mu}{4\pi kh}$, $B = A \ln \frac{4\varkappa}{\gamma\rho^2}$. Since value q_0 is known, after measuring according to curve/graph parameters A and B, it is possible to find the hydroconductivity of layer kh/μ and the relation of $\frac{\varkappa}{\rho^2}$.

It is necessary to keep in mind that the radius of hole, which enters the formulas for an inflow to hole, is not usually equal to a true radius as a result of the fact that, in the first place, the hole reveals layer not to entire power and, in the second place, all surface of hole is opened for the motion of liquid (inadequacy of hole according to degree and character of opening).

Page 49.

Therefore, by knowing the value of $\frac{\varkappa}{\rho^2}$, it is not possible separately to determine $\varkappa$ and ρ_2. For determining the piezoconductivity of layer to conveniently utilize a method of

hydromonitoring, i.e., to investigate the change in pressure in another hole, which did not work up to the torque/moment of the start of the "perturbation" hole. In this case significant dimension is not the radius of hole, but the distance between holes which it is known sufficiently accurate.

We will use for the analysis of the curve of pressure change by the formula, which describes pressure distribution in infinite layer during the work of the hole-source, released with t = 0 with output q_0:

$$p(r, t)-p(r, 0)=-\frac{q_0\mu}{4\pi kh}\operatorname{Ei}\left(-\frac{r^2}{4\varkappa t}\right). \qquad \text{(III.4.2)}$$

After simple transformations formula (III.4.2) can be written in the form:

$$p(r, t)-p(r, 0)=\Delta p=\frac{q_0\mu}{4\pi kh}\int_0^t e^{-\frac{r^2}{4\varkappa t}}\frac{d\tau}{\tau}. \qquad \text{(III.4.3)}$$

The usual method of treatment/working the curves of pressure change in the reacting hole entails the fact that on curve is fixed time of the appearance of any characteristic points - the point of inflection, point of contact of tangency, etc. Conveniently is fixed, for example, point of contact of tangency with the curve of p (t) of straight line, carried out from the origin of coordinates. Let at this point $t = t_1$, $\Delta p = (\Delta p)_1$:

$$\frac{\partial p}{\partial t} = \frac{(\Delta p)_1}{t_1}.$$

By substituting here values dp/dt and Δp from formulas (III.4.2) and (III.4.3), it is possible easily to find that t_1 is determined from equation

$$-\mathrm{Ei}\left(-\frac{r^2}{4\varkappa t_1}\right)=\exp\left(-\frac{r^2}{4\varkappa t_1}\right).$$

. The root of this equation is equal $\frac{r^2}{4\varkappa t_1}=0{,}44,$ whence $\varkappa=0{,}57\frac{r^2}{t_1}.$

2. The applicability of the given simplest methods of the use of inverse problems of filtration theory for the investigation of layers is limited by the conditions under which the hole can be considered as source of constant intensity in infinite uniform layer. When the disturbance/perturbation, caused by the coverage of hole, reaches the boundaries of layer, i.e., through time of the order of the $R^2/\varkappa$, the recovery characteristic of pressure in hole will initiate to be distorted, and through sufficiently long time it emerges to the horizontal asymptote, which corresponds to the stationary distribution of pressure.

Page 50.

Thus, the extent of straight portion in the curve of p (ln t) is limited. At the same time actually on the strength of a series of technical difficulties hole cannot be stopped instantly. Hole usually is closed not on face, i.e., on the boundary of layer, but on surface. Due to the elasticity of liquids and gases, which fill hole, inflow from layer continues still for a while after coverage. Time before leaving to asymptote, obviously, must exceed time of supplementary inflow. Therefore are possible conditions, especially in the holes, arrange/located closely the boundaries of the layer when straight portion in the curve of p(ln t) do not exist.

In addition, supplementary inflow into hole considerably increases the duration of investigation.

The method of treatment/working the recovery characteristics of pressure, free from the indicated drawbacks, was proposed in work [15]. In this method directly are utilized the transformations of Laplace of the curves of the recovery of pressure; therefore it is suitable during an arbitrary change in the output of holes.

The methods of treatment/working the recovery characteristics of pressure outlined above can be obtained as special case of general method. The use of the Laplace transform makes it possible also in many instances to determine according to the recovery characteristics of pressure the character of the heterogeneity of layer (radius of the zone of the lowered/reduced permeability near holes, a distance of impenetrable boundaries etc.).

Let us assume that as a result of measurements in hole are known to dependence of $p(r, t)$ and the $\frac{2\pi kh}{\mu}\rho\frac{\partial p}{\partial r}\Big|_{r=\rho} = q(t)$.

Let us examine the layer, heterogeneous and arbitrary configuration, in which with t = 0 comes into action the hole in nonstationary system. A change in pressure p (r, θ, z, t), calculated off the initial stationary level, satisfies an equation of piezoconductivity

$$\frac{\partial p}{\partial t} - \varkappa \nabla^2 p = 0, \qquad \text{(III.4.4)}$$

and the initial condition p = 0. Let hole ($r = \rho$) satisfy condition $p|_{r=\rho} = p_1(t)$. Furthermore, on the outer edges of layer must be fulfilled the conditions of the type p = 0 with current generators and dp/dn = 0 with the impenetrable sections of boundary. transfer/converting to the transformation of pressure according to Laplace P (r, z, θ, σ), we will obtain that P satisfies an equation

$$\nabla^2 P = \frac{\sigma}{\varkappa} P, \qquad \text{(III.4.5)}$$

to the conditions of $P|_{r=\rho} = P_1(\sigma)$ and conditions of the type P = 0 or dP/dn - 0 on the different sections of the outer edge of layer. Let us introduce now function U = P/P_1 (σ). This function satisfies an equation (III.4.5) and the uniform conditions of the 1st and 2nd kind on outer edges.

Page 51.

On hole U = 1. According to these conditions can be found function U (r, z, θ, σ), not depending on the operating mode of hole.

Let the output of hole will be q (t). We have

$$Q(\sigma)=\int_0^\infty q(t)\,e^{-\sigma t}\,dt=\frac{2\pi kh}{\mu}\,\rho\int_0^\infty e^{-\sigma t}\int_l \frac{\partial p}{\partial r}\,dt\,dl=\frac{2\pi kh}{\mu}\,P_1(\sigma)\int_l \frac{\partial U}{\partial r}\,dl, \tag{III.4.6}$$

where integration it is conducted along the length of the revealed layer part of the hole.

From formula (III.4.6) it follows that the relation

$$\psi(\sigma) = \frac{P_1(\sigma)}{Q(\sigma)} = \left[\frac{2\pi kh}{\mu}\int_0^l \frac{\partial U}{\partial r}\,dl\right]^{-1} \quad \text{(III.4.7)}$$

depends only on the form of the function U and, therefore, it does not depend on the operating mode of hole. The form of the function ϕ (σ) completely is determined by the parameters of layer. When functions p_1 (t) and q (t) are known, functions P_1 (σ) and Q (σ) can be found without difficulties by any by numerical integration. In the form of the function ϕ (σ) in a number of cases it is possible to determine some parameters of layer.

For the case of the instantaneous stoppage of the hole of $Q(\sigma) = \frac{q_0}{\sigma}$; consequently, $-P_1(\sigma) = P^0(\sigma) = \frac{q_0\psi(\sigma)}{\sigma}$. **Thus**, the function of $\psi(\sigma)$ is equal to σP^0 (σ), where P^0 (σ) is a Laplace transform change in pressure p^0 (t) with the instantaneous stoppage of hole.

From formula (III.4.7) it follows (see also §1) that function p

(t) in the general case can be expressed through P^0 (t) with the aid of the integral of Duhamel, if q (t) it is known:

$$p(t)=\int_0^t \frac{dq}{dt} P^0(t-\tau)\,d\tau. \qquad \text{(III.4.8)}$$

In §§2 and 3 present chapters, were obtained the solutions of series of problems of pressure distribution near hole during a abrupt change in the output. The given there functions P (x, y, σ) can be directly utilized for determining the function of $\psi(\sigma) = \frac{\sigma}{q} P$, without finding separately U (x, y, σ). Function U for some of these problems was obtained in the works of G. I. Barenblatt and V. A. Maximov [22] and V. A. Maksmova [75, 76, 3].

Let us examine the simplest example - the hole, which works in uniform infinite layer. Function U takes the form (see §2):

$$U = \frac{K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right)}{K_0\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}, \qquad \text{(III.4.9)}$$

whence

$$\psi(\sigma) = \frac{1}{2\pi\rho\frac{\partial U}{\partial r}\,\frac{kh}{\mu}} = \frac{\mu}{2\pi kh}\,\frac{K_0\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}{\rho\sqrt{\frac{\sigma}{\varkappa}}\,K_1\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}. \qquad \text{(III.4.10)}$$

Page 52.

With practical construction of Laplace transform from p (t) and q (t) it is convenient instead of σ to utilize the parameter $\tau = 1/\sigma$, i.e., to write $P(\tau) = \int\limits_0^\infty p(t)\, e^{-\frac{t}{\tau}}\, d\tau.$ Integrals are calculated for several values of the parameter τ. Actually due to the presence of the factor of $e^{-\frac{t}{\tau}}$ is utilized the only section by the curve of p (t) or q (t) for which t does not exceed 6σ. Very low values τ (it is less than 1-2 min) cannot be undertaken, since on the initial section the curve of p (t) is determined very inaccurately.

Therefore the values of $\rho\sqrt{\frac{\sigma}{\varkappa}}=\frac{\rho}{\sqrt{\varkappa\tau}}$ in all in practice interesting cases are sufficiently low in order that it would be possible to use the representation of functions K_0 (z) and K_1 (z) for the low values of argument (III.3.27). Hence

$$\Psi(\tau)=\psi(\sigma)=-\frac{\mu}{2\pi kh}\ln\frac{\gamma\rho}{2\sqrt{\varkappa\tau}}=\frac{\mu}{4\pi kh}\ln\tau+\frac{\mu}{4\pi kh}\ln\frac{4\varkappa}{\gamma^2\rho^2}. \quad \text{(III.4.11)}$$

. The layer on the basis of formula (III.4.11) are defined from curve/graph P_1/Q from ln τ exactly as the parameters from curve/graph p, ln t at the instantaneous stoppage of hole. The fact that the dependence of $\psi(\ln\tau)$ is rectilinear, makes it possible to limit oneself to calculation of function for very small number of values τ.

In §3 was given the solution of the problem of inflow to hole in layer with the heterogeneity of circular form. From formula (III.3.25) it follows that with small times (large σ) function P (x, y, σ) and, consequently, also $\psi(\sigma)$ takes the same form, as for a uniform layer with the parameters of inner zone. With small σ (large τ) for determining $\psi(\sigma)$ it is possible to use the formula (III.3.31) which gives at the points where $r=a+\rho\ (\rho\ll R,\ \Theta\approx 0)$:

$$\psi(\sigma)=\frac{P\sigma}{q}=\frac{\mu}{2\pi k_1 h}\left(\ln\frac{r}{R}+\frac{k_1-k_2}{k_1+k_2}\ln\frac{R-a}{R}\right)+$$
$$+\frac{\mu}{2\pi k_2 h}\left[\ln\frac{R}{2}\sqrt{\frac{\sigma}{\varkappa_2}}+C+\ln\frac{\varkappa_1}{\varkappa_2}\right]. \qquad \text{(III.4.12)}$$

Formula (III.4.12) can be written otherwise:

$$\Psi(\tau)=\frac{\mu}{4\pi k_2 h}\ln\tau+\frac{\mu}{4\pi k_2 h}\ln\frac{4}{\gamma^2}\frac{\varkappa_2}{\rho^{*2}}, \qquad \text{(III.4.13)}$$

where

$$\rho^*=R\left(\frac{\rho}{R}\right)^{\frac{k_2}{k_1}}\left(\frac{R-a}{R}\right)^{\frac{k_2}{k_1}\cdot\frac{k_1-k_2}{k_1+k_2}}.$$

. Page 53.

In the simplest case when hole is arranged in the center of zone with a radius of R,

$$\rho^* = R\left(\frac{\rho}{R}\right)^{\frac{k_2}{k_1}} = \rho\left(\frac{\rho}{R}\right)^{\frac{k_2}{k_1}-1}$$

(ρ^* - the given radius of hole). Comparing formulas (III.4.11) and (III.4.13), we see that the formula (III.4.13) represents the converted recovery characteristic of pressure in uniform layer with the parameters of outer zone, but in the hole whose radius is equal to ρ^*. It is not difficult to ascertain that ρ^* - an equivalent radius of hole for a stationary inflow in circular layer with the same values kh/μ. The equivalent radius is defined as radius of this ideal hole, stationary inflow to which in layer with the parameters of outer zone is equal to inflow to real hole in heterogeneous layer with the same the pressure differential between the hole and the circular duct.

The examined case of the layer whose permeability near hole (in critical zone) it differs from permeability in exterior, is of great practical interest. The process of drilling and instrumentation of petroleum or gas well and its subsequent work change permeability in critical zone, most frequently they decrease it. for restoration/reduction and increase in the permeability of critical zone, are conducted its different treatment/workings: flushing by acid, formation of cracks by means of the pumping of liquid under high pressure (hydraulic discontinuity/interruption of layer) or explosion (torpedo bombing), etc. The investigation of holes by the method of the recovery of pressure makes it possible to explain the need of conducting such consumption/production/generations, i.e., to determine a reduction in the permeability for critical zone in comparison with the permeability of the remaining part of the layer, and subsequently - to rate/estimate the effectiveness of the works conducted.

The existence of vertical and horizontal cracks near hole leads to the fact that the flow is not radial. However as it was noted in §3, after certain time after the beginning of disturbance/perturbation, the motion of liquid near hole is close to staticnary. The transiency of inflow prononces only at large distances, where the flow can be considered radial. Therefore the

curves of the recovery vosstaovleniya of pressure at great significance τ have an asymptotic straight portion of the type, described by formula (III.4.13). The entering this formula given radius ρ^* is determined from the solution of the problem of stationary inflow to hole in layer with the specified distribution of cracks.

Page 54.

For example for a vertical crack by the length of $2L\rho^* = \frac{L}{2}$, as this follows from the solution, given in §3, No. 3. A series of other problems of inflow to hole with cracks in critical zone is solved by V. A. Maximov [75].

If the value of $\varkappa_2$ is known, for example, on the curves of hydromonitoring, then on the curves of the recovery of pressure it is possible to determine ρ^* - the most important parameter for the calculation of output.

The analysis of the curves of the recovery of pressure makes it

possible also to reveal/detect the existence of the heterogeneity of layer at large distances from hole. The most important problem of this type is the determination of distance of the rectilinear boundary, which divides the ranges of different permeability.

The solution of the corresponding problems of inflow to hole is led in §3. With small τ, i.e., large σ, the function of $\Psi(\tau)$ takes the form, which corresponds to uniform layer. Since always a >> ρ, in curve $\Psi(\ln\tau)$ has the initial straight portion, which corresponds to formula (III.4.11) for a uniform layer. for small σ it is possible to use the formulas (III.3.8), (III.3.12) - (III.3.15), from which it follows

$$P(a+\rho, \sigma) = -\frac{\mu q}{2\pi k_1 \sigma}\left[K_0\left(\sqrt{\frac{\sigma}{\varkappa_1}}\rho\right) + \frac{k_1-k_2}{k_1+k_2} K_0\left(2a\sqrt{\frac{\sigma}{\varkappa_1}}\right) + \Delta_0\right]. \quad \text{(III.4.15)}$$

By using asymptotic expressions for K_0, we will obtain

$$\psi(\sigma)=\frac{P\sigma}{q}=-\frac{\mu}{4\pi k_1}\left[\frac{2k_1}{k_1+k_2}\ln\sigma+\ln\frac{\rho^2}{4\varkappa_1}+\frac{k_1-k_2}{k_1+k_2}\ln\frac{a^2}{2\varkappa_1}+\Delta_0\right] \quad (III.4.16)$$

or

$$\Psi(\tau)=\frac{\mu}{4\pi k_1}\left[\frac{2k_1}{k_1+k_2}\ln\tau+\ln\frac{4}{\gamma^2}\frac{\varkappa_1}{\rho^{*2}}\right], \quad (III.4.17)$$

where the given radius ρ^* is equal to

$$\rho^*=\sqrt{2\rho a}\left(\frac{a}{2\varkappa_1}\right)^{\frac{k_1}{k_1+k_2}}e^{\Delta_0}.$$

From formula (III.4.17) it follows that with large τ the curved $\Psi(\ln\tau)$ has the asymptotic straight portion whose slope/inclination to the axis of the abscissas in $2K_1/(k_1+k_2)$ of

time is more than the slope/inclination of asymptote for a uniform layer. Distance of the interface of the zones of different permeability can be determined by second term of formula (III.4.17), if is known the relation of $\frac{\varkappa_1}{\varkappa_2}$. If the unknown boundary is impermeable, then it is possible to place $k_2 = 0$. In this case, the slope/inclination of asymptote two times is more than in uniform layer, and $\rho^* = \sqrt{2\rho a}$.

Page 55.

Thus, by comparing two straight portions of the converted recovery characteristic of pressure, it is possible to find the values of $\frac{\varkappa}{\rho^2}$ and $\frac{a}{\rho}$. If are known the values of $\varkappa$ or ρ separately, then can be found unknown value a.

The examined, until now, methods of study were related mainly to the case of the monotonically changing output of hole. Sometimes is of interest the study of holes during an alternation in the output. This method was proposed by S. N. Buzinov and I. D. Umrikhin [35]. In this case, in layer, appear the pressure waves, analogous to thermal waves.

Let us assume that the output of hole in uniform infinite layer, beginning with t = 0, changes according to the law

$$q = q_0 \sin \omega t.$$

Thus, is required to find the solution to equation (III.2.1), satisfy zero initial conditions, and also condition

$$\frac{2\pi kh\rho}{\mu}\left(\frac{\partial p}{\partial r}\right)_{r=\rho} = q_0 \sin \omega t. \qquad \text{(III.4.18)}$$

Utilizing a transform of Laplace P (r, σ), we will obtain that P must satisfy an equation (III.2.3) with boundary conditions

$$\frac{2\pi kh}{\mu}\rho\frac{dP}{dr}\bigg|_{r=\rho}=\frac{q_0\omega}{\sigma^2+\omega^2}. \qquad \text{(III.4.19)}$$

Then, utilizing the formulas, obtained in §2, we have

$$P(r,\sigma)=-\frac{q_0\mu K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right)}{2\pi\varkappa h\rho\sqrt{\frac{\sigma}{\varkappa}}K_1\left(\rho\sqrt{\frac{\sigma}{\varkappa}}\right)}\frac{\omega}{\omega^2+\sigma^2}. \qquad \text{(III.4.20)}$$

For the long times when $\rho\sqrt{\frac{\sigma}{\varkappa}}\ll 1,$ expression for P (r, σ) it is simplified:

$$P(r, \sigma) = -\frac{q_0\mu\omega}{2\pi kh}\frac{K_0\left(r\sqrt{\frac{\sigma}{\varkappa}}\right)}{\omega^2+\sigma^2} \equiv -\frac{q\mu\omega}{2\pi kh}\Phi(\sigma). \qquad \text{(III.4.21)}$$

For determining original from image (III.4.21) we will use the theorem about fold, which gives

$$p(t) = \frac{q_0\mu\omega}{2\pi kh}\varphi(t) = \frac{q_0\mu\omega}{2\pi kh}\int_0^t \frac{1}{2\tau}\exp\left(-\frac{r^2}{4\varkappa\tau}\right)\sin\omega(t-\tau)\,d\tau. \qquad \text{(III.4.22)}$$

Us, however, interests a change cf the pressure in of holes e with $r = \rho$. In all in practice interesting cases the period of fluctuation $2\pi/\omega$ considerably exceeds characteristic time.

Page 56.

Then function f (σ) can be approximately presented in the following

form:

$$\Phi(\sigma) = -\frac{1}{2}\frac{\sigma}{\omega^2+\sigma^2}\frac{1}{\sigma}\ln\gamma\sigma - \frac{1}{2}\frac{1}{\omega^2+\sigma^2}\ln\frac{\gamma\rho^2}{4\varkappa}. \quad \text{(III.4.23)}$$

The original of second term is $\frac{1}{2\omega}\ln\frac{\gamma\rho^2}{4\varkappa}\sin\omega t.$ For the calculation of the original of first term, again we will use the theorem about fold and formula $L[\ln t] = -(1/\sigma)\ln\gamma\sigma$. Then

$$\varphi(t) = \frac{1}{2}\int_0^t \ln\tau\cos\omega(t-\tau)d\tau - \frac{1}{2\omega}\ln\frac{\gamma\rho^2}{4\varkappa}\sin\omega t, \quad \text{(III.4.24)}$$

whence it is possible to obtain

$$p(\rho, t) = \frac{q_0\mu}{4\pi kh}\left[\sin\omega t\,\mathrm{Ci}(\omega t) - \cos\omega t\,\mathrm{Si}(\omega t) - \sin\omega t\ln\frac{\gamma^2}{4}\frac{\rho^2\omega}{\varkappa}\right], \quad \text{(III.4.25)}$$

where $Ci(\omega t)$ and $Si(\omega t)$ - integral cosine and sinc (see [40]).

Functions Ci(ωt) and Si(ωt) at great significance ωt (exceeding 1) can be expressed by the asymptotic formulas:

$$\mathrm{Ci}\,(\omega t) \approx \frac{\sin \omega t}{\omega t};\quad \mathrm{Si}\,(\omega t) \approx \frac{\pi}{2} - \frac{\cos \omega t}{\omega t}.$$

Then formula (III.4.25) will take the form:

$$p\,(\rho,\,t) = \frac{q_0\mu}{4\pi kh}\left(-\frac{\pi}{2}\cos\omega t - \ln\frac{\gamma^2}{4}\,\frac{\rho^2\omega}{\varkappa}\sin\omega t + \frac{1}{\omega t}\right). \qquad \text{(III.4.26)}$$

From (III.4.26) it is evident that in the course of time pressure change in hole becomes sinusoidal with period $2\pi/\omega$. The last/latter term in brackets, which expresses the effect of the initial conditions, can be disregarded already through 2-3 periods. With respect to pressure change in hole possible in accordance with formula (III.4.26) on amplitude and phase displacement fluctuation of

pressure (in comparison with the phase of the variation of output) to find the values of hydroconductivity and parameter of $\rho^2/\varkappa$.

~~MICROFICHE HEADER EBR76051860 / cont. / UNCLAS~~

~~MT/ST-76-1860~~

~~EB02~~

~~SUBJECT CODE 1420~~

Pages 57-88.

Chapter IV.

NONLINEAR INVARIANT PROBLEMS OF THE UNSTEADY FILTRATION OF LIQUIDS AND GASES.

§1. General characteristic of the invariant problems of the theory of

unsteady filtration. Self-similar flat free-flow motions on the zero initial level of liquid.

1. General characteristic of the invariant problems of the theory of unsteady filtration. In chapter II, it was shown that the basic problems of the hydrodynamic theory of unsteady filtration lead to the boundary/edge, mixed or initial problems for nonlinear, as a rule, partial differential equations parabolic type. Nonlinearity is generally characteristic for many urgent tasks of the contemporary hydrodynamics: gas dynamics, the theory of waves, theory of the motions of viscous fluid etc. There does not exist at present any general effective analytical methods of the solution of the sufficiently broad classes of the nonlinear tasks of mathematical physics; this entirely is related also to filtration theory. Therefore in filtration theory (as in many other branches of mathematical physics generally and the mechanics of continuous media, in particular) have already have long drawn attention the peculiar particular solutions which are expressed as the functions by one variable. At first these solutions focused on themselves attention only because their obtaining was reduced to the solution to ordinary equations and was represented (especially in pre-computer era) simpler than the solution to equations in the partial derivatives in the general case. During the construction of the different

approximation methods of solution, more common/general/total, these solutions frequently were utilized as standards, making it possible to rate/estimate the accuracy of method. (The approximation methods of analytical solution retain, especially in filtration theory, their value and now, during the wide introduction of machines, since these methods give the analytical formulas, which make it possible to visually trace the effect of the different parameters, but high accuracy in filtration theory does not represent special interest.

Page 58.

These methods will be examined in the following chapter.). In a number of cases of the tasks, described by such solutions, they are of independent interest.

However the main value of such solutions was realized later. It turned out that they are the asymptotic representations of the solutions of the very broad classes of tasks precisely where the detailed structure of boundary and initial conditions ceases to be essential, and these ranges frequently they are most interesting (for example after certain time after the beginning of selection from hole

until the funnel of depression achieves the range of the effect of adjacent hole etc.). Therefore, knowing such sclutions, we actually obtain possibility to judge, at least qualitatively, the behavior of the very broad class of filtration motions.

The important property below solutions in question is their invariance: for some of these solutions - "self-similar" - the distribution of pressures, pressure heads, densities etc. turns out to be always similar to itself, for others - is moved as solid at constant velocity etc. This property is connected with the special character of the tasks, which lead to such solutions. The execution of the determined transforms of the dependent and independent variables leaves equations, the boundary and initial conditions of task by constant/invariable. As they speak in mathematics, these tasks are invariant relative to certain group of continuous transforms. Such tasks are called invariant, they are examined below.

2. Self-similar flat free-flow motions on the zero initial level of liquid. Will be examined below the exact solutions of some nonlinear tasks of unsteady filtration, which are characterized by zero initial condition. The study of this class of motions is of, besides direct, also fundamental interest, since in similar tasks

most strongly is exhibited the substantially nonlinear character of the problem in question and are detected some properties of nonlinear motions, which sharply differ them from the appropriate linear tasks and unavoidably lost during linearization.

For a certainty during the investigation of tasks with zero initial condition, we will examine free-flow flat filtration motions in initially dry soil, bearing in mind that on the strength of that discovered by L. S. Leybenzon analogy (see Chapter II) all results directly they are transferred by the tasks of the isothermal filtration of gas. In this paragraph the solutions stated below were obtained by G. I. Barenblatt [4, 5, 9].

Let us examine the semi-infinite layer, which has from below flat/plane horizontal impenetrable boundary - confining stratum, and from the side of channel - the flat/plane vertical boundary (Fig. IV.1), perpendicular to X-axis and passing through point $x = 0$.

Page 59.

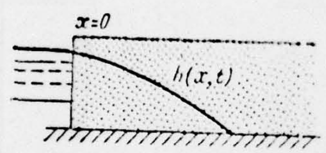


Fig. IV.1.

Let the initial liquid head in layer be equal to zero, and pressure head on the vertical boundary of layer changes according to power law, beginning with the initial torque/moment $t = t_0$:

$$h(0, t) = \sigma (t - t_0)^{\alpha}, \qquad \text{(IV.1.1)}$$

where $\sigma < 0$, a α is certain constant which we will select within limits $-1/2 < \alpha < \infty$. Specifically, the constant α can be equal to zero; in this case the pressure head on boundary instantly takes certain value σ and remains constant.

In the case of the filtration of gas, the formulated task answers the pumping of gas into initially the not filled uniform layer of constant power during a change of the pressure of gas in the initial section of layer $x = 0$ according to the law (IV.1.1). The lines of equal pressure heads will be lines x = const, parallel to the boundary of layer. Thus, pressure head h (x, t) it satisfies an equation

$$\frac{\partial h}{\partial t} = a \frac{\partial^2 h^2}{\partial x^2}; \quad a = \frac{C}{2m} = \frac{k\rho g}{2m\mu}, \qquad \text{(IV.1.2)}$$

being obtained from the common/general/total equation of Boussinesq (II.3.20) for these geometric conditions of task, and also to boundary condition (IV.1.1), to the initial condition and condition at infinity:

$$h(x, t_0) = h(\infty, t) = 0. \quad \text{(IV.1.3)}$$

Head at certain point of layer h depends on the following arguments: coordinate x, the time, passed from the beginning of process $t - t_0$ [on the strength of the uniformity of equation (IV.1.2) on time the pressure head will depend only on difference $t - t_0$, but not on values of t and t_0 individually], coefficients a and σ and constant α. Introducing for a convenience the independent dimensionality of pressure head (this possibly, since for the task in question it is unessential, that the dimensionality of length and pressure head are identical) [1], we will obtain the dimensionality of these arguments in the following form:

$$[a] = [h]^{-1} L^2 T^{-1}; \quad [t - t_0] = T; \quad [x] = L; \quad [\sigma] = [h]\, T^{-\alpha}, \quad \text{(IV.1.4)}$$

where through [h], L and T are designated with respect to the dimensionality of pressure head, length and time; constant α is

dimensionless.

FOOTNOTE [1]. Actually, in this case it would be possible instead of pressure head h to introduce proportional to it pressure of the bottom of layer $h\rho g$, which would not be reflected in remaining lining/calculations. ENDFOOTNOTE.

From the arguments on which depends the liquid head, it is possible to compose only two independent dimensionless combinations:

$$\xi = x\sqrt{\frac{\alpha+1}{a\sigma(t-t_0)^{\alpha+1}}};\ \alpha. \qquad \text{(IV.1.5)}$$

. Page 60.

On the strength of the π-theorem of dimensional analysis

expression for a pressure head can be presented in the form of the product of the combination of the determining parameters, which has the dimensionality of pressure head [as it it is possible to take $\sigma(t-t_0)^\alpha$], for dimensionless function of dimensionless combination (IV.1.5). We have thus

$$h=\sigma(t-t_0)^\alpha f(\xi,\lambda);\quad \lambda=\alpha/(1+\alpha), \tag{IV.1.6}$$

where f - the dimensionless function, and the parameter λ is introduced instead of the parameter α for convenience in the subsequent presentation. It is obvious that λ it lie/rests at interval $-1 < \lambda < 1$. We have, further, on the strength of (IV.1.6)

$$\frac{\partial h}{\partial t}=\alpha\sigma(t-t_0)^{\alpha-1}f(\xi,\lambda)-\sigma(t-t_0)^\alpha\frac{\alpha+1}{2}x\sqrt{\frac{\alpha+1}{a\sigma(t-t_0)^{\alpha+1}}}\frac{df}{d\xi};$$

$$\frac{\partial^2h^2}{\partial x^2}=\frac{\sigma^2(t-t_0)^{2\alpha}(\alpha+1)}{a\sigma(t-t_0)^{\alpha+1}}\frac{d^2f^2}{d\xi^2}.$$

Substituting these relationship/ratics in equation (IV.1.2)

and simplifying, we obtain for function f the ordinary differential equation:

$$\frac{d^2f}{d\xi^2} + \frac{1}{2}\xi\frac{df}{d\xi} - \lambda f = 0. \qquad \text{(IV.1.7)}$$

After the substitution of expression (IV.1.6) into boundary condition (IV.1.1) and condition (IV.1.3) we obtain for function f (ξ, λ) the boundary conditions:

$$f(0, \lambda) = 1; \qquad \text{(IV.1.8)}$$

$$f(\infty, \lambda) = 0. \qquad \text{(IV.1.9)}$$

The pressure head and the volumetric flow (flow rate) of

ground water must be the continuous functions x and t. Utilizing a law of darcys, we have for the flow rate, per unit the unit of the width of layer, expression

$$-Ch\frac{\partial h}{\partial x}=-\frac{C}{2}\frac{\partial h^2}{\partial x}=-\frac{C\sigma^{1/2}(t-t_0)^{\frac{3\alpha-1}{2}}}{2a}\sqrt{a+1}\,\frac{df^2}{d\xi}. \quad \text{(IV.1.10)}$$

Thus, from the requirement for the continuity of flow rate follows the continuity of function $df^2/d\xi$.

With continuous function f (ξ) and f $\neq$ 0, requirement for the continuity of function $df^2/d\xi = 2fdf/d\xi$ coincides with the requirement for the continuity of derivative $df/d\xi$. However, with f = 0 of continuity $df^2/d\xi$, continuity $df/d\xi$ does not escape/ensue. On the contrary as let us see further, the unknown function f (ξ, λ) it has at the point where f becomes zero, discontinuity/interruption of first-order derivative.

Page 61.

Condition (IV.1.9) to conveniently lead tc ancther form. Let us multiply both parts of the fundamental equaticn (IV.1.2) by x and integrate over x from zero tc. As a result we will obtain

$$\int_0^\infty x\frac{\partial h}{\partial t}dx = \frac{d}{dt}\int_0^\infty xh(x, t)\,dx = a\int_0^\infty x\frac{\partial^2 h^2}{\partial x^2}dx =$$
$$= a\left(x\frac{\partial h^2}{\partial x}\right)_{x=0}^{x=\infty} + a\,[h^2(0, t) - h^2(\infty, t)].$$

It is obvious that $\partial h^2/\partial x$ it vanishes with $x \to \infty$ faster than x^{-1}, ctherwise of h, it would not apprcach zerc with $x \to \infty$. Utilizing this fact and a condition at infinity (IV.1.3), we obtain

AD-A044 774 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 13/11
THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS (TEORIY--ETC(U)
JAN 77 G I BARENBLATT, V M YENTOV, V M RYZHIK
UNCLASSIFIED FTD-ID(RS)T-1860-76-PT-1 NL

3 OF 5
AD
A044774

$$\frac{d}{dt}\int_0^\infty xh(x, t)\,dx = ah^2(0, t).$$

Integrating this relationship/ratio within limits from $t = t_0$ to t and utilizing a boundary condition (IV.1.1) and a representation cf solution (IV.1.6), we have

$$\int_0^\infty xh(x,t)\,dx=\frac{a\sigma^2(t-t_0)^{2\alpha+1}}{\alpha+1}\int_0^\infty \xi f(\xi,\lambda)\,d\xi=$$

$$=a\int_{t_0}^{t} h^2(0,t)\,dt=\frac{a\sigma^2(t-t_0)^{2\alpha+1}}{2\alpha+1}$$

(recall that we consider α satisfying inequality $-1/2<\alpha<\infty$), whence we obtain the unknown condition in the form

$$\int_0^\infty \xi f(\xi,\lambda)\,d\xi=\frac{1+\alpha}{1+2\alpha}=\frac{1}{1+\lambda}. \qquad \text{(IV.1.11)}$$

In the which interests us range of change α and λ the right side (IV.1.11) is final and positive.

3. Study of the integral curves of ordinary differential equation. Thus, the solution of the problem in question came to the solution to the ordinary differential equation of the second order (IV.1.7) under the conditions (IV.1.8) and (IV.1.11), continuous and having continuous derivative of square. Let us note that the equation (IV.1.7) is invariant relative to transformation group

$$\Phi(\xi, \mu) = \mu^{-2} f(\mu\xi, \lambda), \qquad \text{(IV.1.12)}$$

i.e. if f (ξ, μ) satisfies an equation (IV.1.7), then also Φ (ξ, μ) satisfies this equation with arbitrary positive μ.

Page 62.

This property of equation makes it possible to lower its order. Let us assume according to general rule (for example, see [53, page 93])

$$f(\xi, \lambda) = \xi^2 \varphi(\eta, \lambda), \quad \eta = \ln \xi, \qquad \text{(IV.1.13)}$$

then equation (IV.1.7) will come to second order equation relative to

the function of $\varphi(\eta)$, not containing the independent alterrating/variable η:

$$\varphi\varphi'' + 6\varphi^2 + 7\varphi\varphi' + \varphi'^2 + \frac{1}{2}(1-\lambda)\varphi + \frac{1}{4}\varphi' = 0. \qquad \text{(IV.1.14)}$$

By set/assuming further

$$\frac{d\varphi}{d\eta} = \psi \qquad \text{(IV.1.15)}$$

and by accepting ϕ as independent variable, we will obtain for function ψ first-order equation:

$$\frac{d\psi}{d\varphi} = -\frac{1}{\varphi\psi}\left[6\varphi^2 + 7\varphi\psi + \psi^2 + \frac{1-\lambda}{2}\varphi + \frac{1}{4}\psi\right]. \qquad \text{(IV.1.16)}$$

The investigation of this equation is carried out by usual method (for example, see V. V. Stepanov's book [111]). Since,

obviously, the pressure head is knowingly nonnegative, function f and, therefore, function ϕ are also nonnegative, so that the which interests us range of plane $\varphi\psi$ is the right half-plane (see Fig. IV.2).

Near axis ψ (i.e. that, where ϕ is small even $\psi \gg \phi$) equation (IV.1.16) is record/written as follows:

$$\frac{d\psi}{d\varphi} = -\frac{1}{\varphi}\left(\psi + \frac{1}{4}\right) + 0(1). \qquad \text{(IV.1.17)}$$

. That means with small ϕ and $\psi > -1/4$ the integral curves have large negative slope/inclination, with $\psi < -1/4$ - large positive slope/inclination. By integrating equation (IV.1.17), we will obtain that near axis ψ the integral curves are represented by formula

$$\psi = \frac{C}{\varphi} - \frac{1}{4} + 0(\varphi), \qquad \text{(IV.1.18)}$$

where C is the constant of integration, different for different integral curves. For the study of the behavior of integral curves in the vicinity of the origin of coordinates, let us conduct through the beginning are direct/straight $\psi = m\phi$ let us examine the behavior of integral curves on these straight lines near beginning. We have on straight line $\psi = m\phi$ near beginning

$$\frac{d\psi}{d\varphi} = -\frac{1}{4m\varphi}[m + 2(1-\lambda)] + 0(1), \qquad \text{(IV.1.19)}$$

so that with $m > m_0 = -2(1-\lambda) = -2/(1+\alpha)$ a slope/inclination of integral curves is great and negative, with $m < m_0$ is great and positive.

Page 63.

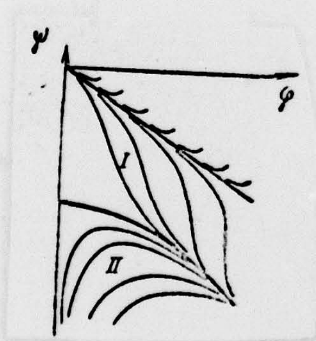


Fig. IV.2.

As it is not difficult to see, with positive ϕ and , i.e., in the first quadrant, the slope/inclination of integral curves is negative. Near axis ϕ, i.e., with small $\psi << \phi$, the equation (IV.1.16) is represented in the form:

$$\frac{d\psi}{d\varphi} = -\frac{1}{\psi}\left[6\varphi + \frac{1-\lambda}{2}\right] + 0(1), \qquad \text{(IV.1.20)}$$

therefore near this axis the slope/inclination of integral curves reverses the sign, going to infinity. Thus, integral curve first-order equations (IV.1.16) take the form, depicted on Fig. IV.2. Depending on that, is positive C or negatively, these integral curves are divide/marked off into two classes: I and II. Equation (IV.1.18) shows that not one of the integral curves I class (C > 0) and not one of the integral curves II class (C < 0) not transverse axis ψ at the end point. The curves I class near the origin of coordinates approach agreement with straight line $\psi = m_0\phi = -2\phi/\alpha+1$, so that near the origin of the coordinates of plane $\varphi\psi$ these curves satisfy an equation

$$\frac{d\varphi}{d\eta} = -\frac{2}{\alpha+1}\varphi + 0(\varphi). \qquad \text{(IV.1.21)}$$

Integrating this equation, we obtain

$$\ln\varphi=-\frac{2}{\alpha+1}\eta+\ln D+\ldots=-\frac{2}{\alpha+1}\ln\xi+\ln D+\ldots,$$

$$\varphi=D\xi^{-\frac{2}{\alpha+1}}, \tag{IV.1.22}$$

where D, the constant of integration, and, dots they mean the negligible low values.

From (IV.1.22) it is evident that with approach along the integral curve in question at the beginning of coordinates, i.e., with $\phi \to 0$, ξ it approaches infinity. Returning to alternating/variable f and ξ, we obtain, that integral curve second order equations (IV.1.7), that correspond to integral curves I class of first-order equation, with $\xi \to \infty$ satisfy relationship/ratio

$$f = D\xi^{2\alpha} + o(\xi^{2\alpha}). \qquad \text{(IV.1.23)}$$

Further, with small ϕ for integral curves II class we have

$$\varphi\psi = \varphi \frac{d\varphi}{d\eta} = C + 0(\varphi) \qquad (C < 0). \qquad \text{(IV.1.24)}$$

Page 64.

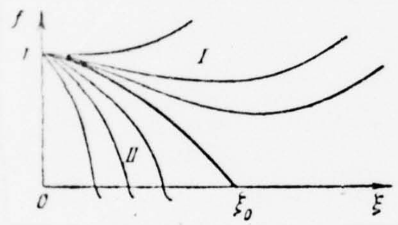


Fig. IV.3.

By integrating this equation, we will obtain relationship/ratio

$$\varphi^2 = 2C\eta + E + 0(\varphi^2),$$

which it shows that the η remains final with $\phi \to 0$, i.e., with f 0 for the appropriate integral curves of second order equation ξ, it remains final. Having this in form and transfer/converting in relationship/ratio (IV.1.24) to alternating variable f, ξ, we obtain, that with small f the corresponding integral curves II class of second order equation (IV.1.7) satisfy relationship/ratio

$$\frac{df^2}{d\xi} = 2C\xi^3 + 0(f). \qquad \text{(IV.1.25)}$$

Integral curves I and II classes of the equation of the first order (IV.1.16) are divided with the integral curve, which corresponds by C = 0, which near axis ψ is represented by equation

$$\psi = \frac{d\varphi}{d\eta} = -\frac{1}{4} + 0(\varphi). \quad \text{(IV.1.26)}$$

Transfer/converting to alternating/variable f and ξ, we obtain, that the dividing curve with small f satisfies relationship/ratio

$$\frac{df}{d\xi} = -\frac{1}{4}\xi + 0(f). \quad \text{(IV.1.27)}$$

The systems of the integral curves of second order equation (IV.1.7), that take different values with ξ = 0, they are obtained

one of another by similarity transformation (IV.1.12). Thus, summing up everything said, we obtain, that integral curve equations (IV.1.7), which satisfy condition (IV.1.8), are arrange/located as follows (Fig. IV.3). The curves I class with $\xi \to \infty$ change according to the law of $f = D\xi^{2\lambda}$ ($D \neq 0$ - the constant, different for different curves), whereupon not one of these curves not at one point, it clear and not the transverse axis of abscissas. It is obvious that not one of these curves is unknown, since not one of them satisfies condition (IV.1.11) - for each of them the integral of $\int_0^\infty \xi f(\xi, \lambda) d\xi$ diverges. By exception/elimination is the case $\alpha = 0$ (examined/considered is below), for which all the curves I class have horizontal asymptotes. Figure IV.3 depicts the case $\alpha >$ as 0. Remaining integral curves (curves II class) the transverse axis of abscissas at the end points, they approaching the axis of abscissas at right angles, since for each of these curves the relationship/ratio (IV.1.25) gives with small f

$$\frac{df}{d\xi} = \frac{C}{f}\xi_C^2 + 0(1). \qquad (IV.1.28)$$

. Page 65.

Here C < 0 are the constant, which corresponds to the integral curve in question II class, the transverse axis of abscissas at point with the coordinate of ξ_C. The dividing curve approaches the axis of abscissas at point with the coordinate of aaaa. The dividing curve approaches the axis of abscissas at point $\xi = \xi_0$ at certain angle of aaaa. On the strength of (IV.1.27) this angle is determined by relationship/ratio

$$\operatorname{tg} \nu = -\frac{1}{4}\xi_0.$$

. Since the liquid head according to the physical considerations cannot be negative [1], it is clear that the unknown function f (ξ, λ) must by any form be combined from the integral curves of equation (IV.1.7), not belonging to I to class, in their that part where these curves are arrange/located above the axis of abscissas, and from the very axis of abscissas.

FOOTNOTE [1]. Mathematically this is the consequence of the fact that for equation (IV.1.2) is valid the principle of the maximum, in accordance with which solution cannot turn out to be negative under positive initial and boundary conditions. ENDFOOTNOTE.

However, if we compose function f (ξ, λ) in such a way that it would be represented by segment of certain curve II class up to the point of the ξ_C of the intersection of this curve with the axis of abscissas, and the further very axis of abscissas, then the obtained function at the point of $\xi = \xi_C$ will have a derivative discontinuity of square. In fact, during approach/approximation to the point of intersection of $\xi = \xi_C$ to the right, where function f (ξ, λ) it is represented by the axis of abscissas, we obtain that the $(df^2/d\xi)_{\xi=\xi_C+0} = 0$, since with $df^2/d\xi$ is identically equal to zero of $\xi > \xi_C$. During approach/approximation to the point of intersection of $\xi = \xi_C$ to the left, where function f (ξ, λ) are represented by certain curve II class, we obtain on the strength of relationship/ratio (IV.1.25)

$$\left(\frac{df^2}{d\xi}\right)_{\xi=\xi_C-0} = 2C\xi_C^3 \neq 0. \qquad \text{(IV.1.29)}$$

The discontinuity/interruption of value $df^2/d\xi$ corresponds to the discontinuity/interruption of fluid flow, which contradicts the

formulation of the problem. Therefore not one of the functions f (ξ, λ), the being obtained by the combination pointed out above integral curves II class with $C \neq 0$ and axis of abscissas, do not befit.

By the unknown curve of equation (IV.1.7), that satisfies condition (IV.1.8), by the continuous and possessing continuous derivative of square, will be the curved, consisting of cut integral curve, which divides the curves I and II classes, up to the intersection of it with number system the axis of abscissas at certain point ξ_0, and the axis intercept of abscissas $\xi \geq \xi_0$.

Function itself is continuous by construction; is checked the continuity of derivative of square at point of intersection $\xi = \xi_0$ (in remaining points this continuity does not cause doubts, since the integral curve consists of two sections of smooth curves). With approach to point $\xi = \xi_0$ to the right, where integral curve is represented by the axis of abscissas, the limit of $(df^2/d\xi)_{\xi=\xi_0+0}$ is equal to zero. With approach to point $\xi = \xi_0$ to the left the limit is equal $(df^2/d\xi)_{\xi=\xi_0-0} = 2\,(fdf/d\xi)_{\xi=\xi_0-0}$ and on the strength of (IV.1.27) is equal, i.e., $^1/_4\ (\xi f)_{\xi=\xi_0} = 0$.

Page 66.

Thus, for that which was constructed the curved derivative $df^2/d\xi$ is continuous.

Let us show now that the constructed function satisfies condition (IV.1.11). Let us multiply both parts of the equation (IV.1.7) on ξ and integrate within limits from $\xi = 0$ to $\xi = \infty$ (or, that the same, to $\xi = \xi_0$, since with $\xi \geqslant \xi_0\ f(\xi, \lambda) \equiv 0$). We will obtain

$$-\lambda \int_0^{\xi_0} \xi f(\xi)\, d\xi + \frac{1}{2} \int_0^{\xi_0} \xi^2 \frac{df}{d\xi}\, d\xi + \int_0^{\xi_0} \xi \frac{d^2 f^2}{d\xi^2} = 0. \qquad \text{(IV.1.30)}$$

But on the strength of continuity f and $df^2/d\xi$ we have

$$\int_0^{\xi_0} \xi^2 \frac{df}{d\xi} d\xi = \xi^2 f \Big|_0^{\xi_0} - 2\int_0^{\xi_0} \xi f(\xi, \lambda) d\xi = -2\int_0^{\xi_0} \xi f(\xi, \lambda) d\xi;$$

$$\int_0^{\xi_0} \xi \frac{d^2 f^2}{d\xi^2} d\xi = \xi \frac{df^2}{d\xi}\Big|_0^{\xi_0} - \int_0^{\xi_0} \frac{df^2}{d\xi} d\xi = f^2(0, \lambda) = 1,$$

whence and from (IV.1.30) obtain

$$\int_0^{\infty} \xi f(\xi, \lambda) d\xi = \int_0^{\xi_0} \xi f(\xi, \lambda) d\xi = \frac{1}{1+\lambda}, \qquad \text{(IV.1.31)}$$

i.e. function f (ξ, λ) it satisfies the condition (IV.1.11), Q. E. D.

Thus, function f (ξ, λ) it differs from zero only with $\xi < \xi_0$,

and with ξ :ml ξ_0 it is identically equal tc zero. It goes without saying that value ξ_0 depends on the parameter λ. At point $\xi = \xi_0$, function f (ξ, λ) has a discontinuity/interruption of first-order derivative [1].

FCCTNCTE [1]. Thus, the cbtained sclutior h (x, t) equation in the partial derivatives (IV.1.2) it has a derivative discontinuity dh/dx and therefore is not the soluticn to this equation in classical sense, but represents the generalized sclution to this equation according to S. P. Sobolyov [107]. ENDFCOTNOTE.

From the requirement fcr continuity f and $df^2/d\xi$ and for the uniqueness theorem of the sclution to differential equation, it follows that during the composition of functior f (ξ, λ) the cementing cf the different integral curves cf equation (IV.1.7) can be produced only at the points where f = 0, whence directly escape/ensues uniqueness the constructed by us function, i.e., the uniqueness of self-similar solution [2].

FCCTNCTE [2]. Relative to the proofs of uniqueness in self-similar ncnlinear problems it is possitle to make the fcllowing generality.

The given reasonings (and analogous reasonings for other tasks), which prove uniqueness the solutions tc boundary-value problem for ordinary equation, can serve only as prcof of the uniqueness of the self-similar soluticns of the problems in question. The very prcof of the self-similarity of solutions, emanating from the appropriate settings of boundary-value problems and based on π-theorem, rests on assumption about the fact that the sclution can depend only on the dimensional parameters, which enter the equations and the boundary conditions of task (in other words, it is assumed that the system of the specifying parameters is complete). Thus, automatically are eliminated all the possible families of soluticns, which are characterized even any dimensional parameters. It is possible to give the elementary example, which well illustrates this fact. Solution to the equation of the thermal conductivity cf $a^2u_{xx} = u_t$ under the conditions u (0, t) = U = const and u (∞, t) = 0 knowingly not singularly; however as nct difficult tc show, the self-similar solution of this problem singularly. The complete proof of the uniqueness of solution in the natural fcr the tasks in question class of functions requires even for the self-similar problems of the enlistment of supplementary consideraticns. ENDFOOTNOTE.

Page 67.

4. Effective calculation of function f (ξ, λ). For the effective calculation of function f (ξ, λ) to inexpediently turn to the integration of first-order equation (IV.1.16). To conveniently enter as follows.

Let us construct the solution of the $\Phi(\xi, \lambda)$ of second order equation (IV.1.7), that turns with ξ = 1 into zero and having at this point final first-order derivative, i.e., corresponding to the dividing integral curve, passing through the point ξ = 1. On the strength of equality (IV.1.27), regardless of the fact, is satisfied condition (IV.1.8) or not, this derivative is equal - 1/4.

In ξ < 1 solution of $\Phi(\xi, \lambda)$ is represented by the rapidly converging series

$$\Phi(\xi, \lambda)=\frac{1}{4}(1-\xi)+C_1(1-\xi)^2+C_2(1-\xi)^3+\ldots, \quad \text{(IV.1.32)}$$

where

$$C=\frac{2\lambda-1}{16},\quad C_2=\frac{2}{9}C_1(\lambda-1-12C_1),\ C_3=C_2\left(\frac{2\lambda-3}{16}-5C_1\right). \qquad \text{(IV.1.33)}$$

. A series (IV.1.32) fast enough converges on all cut $0 \leq \xi \leq 1$; however, for the calculation of $\Phi(\xi, \lambda)$ at the points, close to $\xi = 1$. Summarizing a series (IV.1.32) with $\xi = 0$ or calculating $\Phi(0, \lambda)$ by numerical integration, it is possible to obtain $\Phi(0, \lambda) = N(\lambda)$, where $N(\lambda)$ is the positive number, not equal to one. Thus, the function of $\Psi(\xi, \lambda)$, equal $\Phi(\xi, \lambda)$ with $\xi \leq 1$ and identically equal to zero with $\xi > 1$ is continuous and has continuous derivative of square, satisfies an equation (IV.1.7) and condition at infinity (IV.1.9), but condition (IV.1.8) it does not satisfy.

For obtaining the unknown solution, let us recall that the function

$$f(\xi, \lambda)=\frac{1}{\mu^2}\Psi(\mu\xi, \lambda) \qquad \text{(IV.1.34)}$$

also satisfies an equation (IV.1.7) with arbitrary $\mu > 0$ and possesses the necessary properties of continuity. Let us select now $\mu = M_0$ by such form, in order to function f (ξ, λ) satisfied also condition (IV.1.8), then the obtained function f (ξ, λ) it will satisfy all conditions, superimposed on the unknown solution. We have

$$f(0, \lambda) = 1 = \frac{1}{\mu_0^2}\Phi(0, \lambda) = \frac{1}{\mu_0^2}N(\lambda), \qquad \text{(IV.1.35)}$$

whence we obtain

$$\mu_0 = \sqrt{N(\lambda)}. \qquad \text{(IV.1.36)}$$

Pages 68 and 69.

Table IV.1.

ξ/ξ^* \ λ	0,00	0,05	0,10	0,15	0,20	0,25	0,30	0,35	0,40	0,45
0,1	0,9257	0,9222	0,9190	0,9161	0,9133	0,9107	0,9083	0,9060	0,9034	0,9019
0,2	0,8461	0,8399	0,8341	0 8288	0,8239	0,8192	0,8149	0,8108	0,8070	0,8034
0,3	0,7610	0,7528	0,7452	0,7382	0,7316	0,7255	0,7198	0,7144	0,7093	0,7045
0,4	0,6702	0,6608	0,6521	0,6440	0,6365	0,6294	0,6228	0,6166	0,6107	0,6052
0,5	0,5738	0,5639	0,5547	0,5463	0,5383	0,5309	0,5240	0,5174	0,5113	0,5055
0,6	0,4714	0,4618	0,4530	0,4448	0,4372	0,4300	0,4232	0,4169	0,4110	0,4053
0,7	0,3629	0,3545	0,3468	0,3395	0,3328	0,3265	0,3205	0,3149	0,3097	0,3047
0,8	0,2483	0,2419	0,2359	0,2304	0,2252	0,2203	0,2158	0,2115	0,2074	0,2036
0,9	0,1273	0,1237	0,1204	0,1172	0,1143	0,1115	0.1090	0,1065	0,1042	0,1021
$\xi^*(\lambda)$	2,286	2,250	2,216	2,185	2,154	2,126	2,098	2,072	2,047	2,023

Table IV.1 Continuation

0.50	0,55	0,60	0,65	0,70	0,75	0,80	0.85	0,90	0,95	1,00
0,9000	0,8982	0,8965	0,8949	0,8933	0,8918	0,8904	0,8890	0,8876	0,8863	0,8850
0,8000	0,7968	0,7937	0,7908	0,7880	0,7853	0,7828	0,7803	0,7779	0,7756	0,7735
0,7000	0,6957	0,6916	0,6878	0,6841	0,6806	0,6772	0,6739	0,6708	0,6677	0,6647
0,6000	0,5951	0,5904	0,5859	0,5816	0,5776	0,5737	0,5699	0,5663	0,5628	0,5594
0,5000	0,4948	0,4899	0,4852	0,4807	0,4764	0,4723	0,4683	0,4645	0,4609	0,4573
0,4000	0,3950	0,3902	0,3856	0,3812	0,3771	0,3731	0,3693	0,3656	0,3621	0,3586
0,3000	0,2955	0,2913	0,2873	0,2834	0,2797	0,2762	0,2728	0,2696	0,2665	0,2634
0,2000	0,1966	0,1933	0,1902	0,1872	0,1843	0,1817	0,1791	0,1765	0,1742	0,1718
0,1000	0,0980	0,0962	0,0944	0,0927	0,0911	0,0896	0,0881	0,0867	0,0853	0,0840
2,000	1,978	1,957	1,936	1,916	1,897	1,879	1,861	1,844	1,827	1,810

Value ξ_0 (beginning with which f (ξ, λ) of $\equiv$ 0) is obtained if to consider that the Ψ (μ_0, ξ, λ) of $\equiv$ 0 when $\mu_0 \xi \geqslant 1$, whence, and also from (IV.1.34) it follows that

$$\xi_0 = \frac{1}{\mu_0} = \frac{1}{\sqrt{\bar{N}(\lambda)}}. \qquad \text{(IV.1.37)}$$

The function of Φ (ξ, λ) and, therefore, of Ψ (ξ, λ) is determined by the addition of a series (IV.1.32) or by numerical integration; knowing $\mu = \mu_0$, it is possible thus to calculate f (ξ, λ) according to formula (IV.1.34). The results of calculations f (ξ, λ) for a series of values λ, are brought in Table IV.1 and IV.2 and are given in Fig. IV.4, but in Fig. IV.5 are represented to function $\xi_0(\lambda)$ and to $M(\lambda) = df^2(0, \lambda)/d\xi$.

We see that the curves of f (ξ, λ), appropriate $\lambda > 1/2$, are directed by concavity upward; curve, appropriate $\lambda = 1/2$, it is broken line, comprised of two straight lines; with $\lambda < 1/2$ curved f

(ξ, λ) are directed by concavity down, whereupon up to the function, which corresponds $\lambda = -1/2$, the derivative $f'(0, \lambda)$ is negative. To value $\lambda = -1/2$ corresponds function

$$f\left(\xi, -\frac{1}{2}\right) = \begin{cases} 1 - \frac{1}{8}\xi^2 & (0 \leqslant \xi \leqslant \sqrt{8}) \\ 0 & (\xi \geqslant \sqrt{8}), \end{cases} \qquad \text{(IV.1.38}$$

having $f'(0, -1/2) = 0$. With $\lambda < -1/2$ $f'(0, \lambda)$ it is positive.

Page 70.

Table IV.2.

λ	$M(\lambda)$	λ	$M(\lambda)$	λ	$M(\lambda)$
0,00	0,6276	0,35	0,9010	0,70	1,120
0,05	0,6714	0,40	0,9349	0,75	1,149
0,10	0,7134	0,45	0,9680	0,80	1,176
0,15	0,7538	0,50	1,0000	0,85	1,203
0,20	0,7925	0,55	1,031	0,90	1,229
0,25	0,8299	0,60	1,062	0,95	1,255
0,30	0,8661	0,65	1,091	1,00	1,280

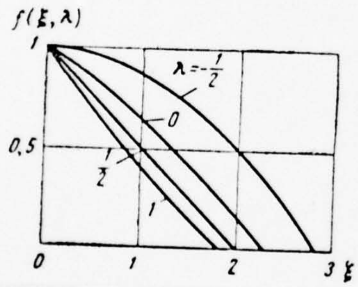


Fig. IV.4.

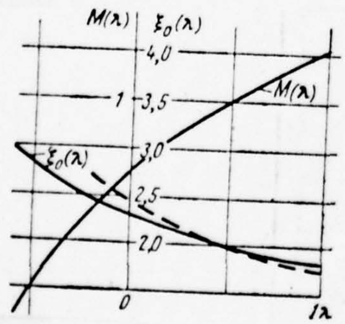


Fig. IV.5.

The function ξ_0 (λ) monotonically grow/rises with decrease λ, strive for infinity with λ, that approaches -1 (solution, which corresponds λ = -1, there will be examined below).

5. Fundamental characteristics of the investigated self-similar solutions. Transfer/converting from function f (ξ, λ) to liquid head h, we obtain, that the liquid head differs from zero at each point in time only in certain final part of the range of the porous medium in question, whereupon the size/dimension of this range increases in the course of time. The finiteness of the velocity of the propagation of the front/leading boundary of the disturbed range is characteristic for the circle of the tasks in question, which correspond zero initial condition; it essentially differs the formulation of the problem of flat free-flow motions from the tasks, connected with the classical linear parabolic equations, for which, as is known, it occurs the infinite velocity of propagation of the leading edge of the disturbed range.

This special feature/peculiarity was reveal/detected for the first time in works Ya. of B. Zeldovich and A. S. Kompaneyts [50 [and G. I. Barenblatt [4] by the investigation of different self-similar solutions.

Page 71.

In the work of G. I. Barenblatt and M. I. Vishik [16] was given the proof of the finiteness of the velocity of propagation of the front/leading boundary of the disturbed range for the tasks of the flat free-flow motions (and also the broad class of more common/general/total tasks), corresponding to the initial distributions of liquid head, identically equal to zero outside certain finite domain.

The coordinate of the driving leading edge of liquid for the self-similar motions in question is expressed by formula

$$x_0(t)=\xi_0\sqrt{\frac{a\sigma\,(t-t_0)^{\alpha+1}}{\alpha+1}} \qquad (IV.1.39)$$

(since leading edge it corresponds $\xi = \xi_0$; recall that the parameters

α and λ are connected between themselves by relationship/ratio $\lambda = \alpha/(\alpha + 1)$). The velocity of propagation of leading edge v_0 is represented by relationship/ratio

$$v_0 = \frac{1}{2} \xi_0 \sqrt{a\sigma (t - t_0)^{\alpha-1} (\alpha + 1)}.$$

In particular, when pressure head on the boundary of layer is constant, i.e., $\alpha = 0$, then

$$x_0(t) = 2{,}286 \sqrt{a\sigma (t - t_0)}; \quad v_0 = 1{,}143 \sqrt{\frac{a\sigma}{t - t_0}}. \qquad \text{(IV.1.40)}$$

Further, for the total volume of liquid in layer M on the

basis of equations (IV.1.5) and (IV.1.6) is obtained following of the expression:

$$M=\int_0^\infty mh(x,t)\,dx=\frac{ma^{1/2}\sigma^{2/2}(t-t_0)^{\frac{3\alpha}{2}+\frac{1}{2}}}{\sqrt{\alpha+1}}\int_0^{\xi_0} f(\xi,\lambda)\,d\xi, \tag{IV.1.41}$$

a for a fluid flow with x = 0, i.e., for the velocity of influx of liquid into layer, on the strength of (IV.1.10) are an expression

$$-\frac{1}{2}C\left(\frac{\partial h^2}{\partial x}\right)_{x=0}=-\frac{C\sigma^{3/2}(t-t_0)^{\frac{3\alpha}{2}-\frac{1}{2}}}{2a}\sqrt{\alpha+1}\left(\frac{df^2}{d\xi}\right)_{\xi=0}. \tag{IV.1.42}$$

Integrating both parts of the equation (IV.1.7) for ξ from ξ = 0 to $\xi = \infty$ or, that nevertheless, to $\xi = \xi_0$, since f (ξ, λ) of $\equiv$ 0

with $\xi \geqslant \xi_0$, we obtain

$$\int_0^{\xi_0} f(\xi, \lambda)\, d\xi = -\frac{2}{1+2\lambda} \frac{df^2}{d\xi}\Big|_{\xi=0}, \qquad \text{(IV.1.43)}$$

so that formula (IV.1.41) is reduced to the form:

$$M = -\frac{2ma^{1/4}\sigma^{3/4}(t-t_0)^{\frac{3\alpha+1}{2}}\sqrt{1+\alpha}}{1+2\alpha}. \qquad \text{(IV.1.44)}$$

~~MICROFICHE HEADER EBR76061860 / cont. / UNCLAS~~

~~MT/ST-76-1860~~

~~DE03~~

~~SUBJECT CODE 142D~~

Page 72.

Thus, the preceding/previous relationship/ratios show that the solutions, which correspond to $0 < \alpha < \infty$, i.e., $0 < \lambda$, 1, answer an increase in the liquid head on boundary and in the total amount of liquid in layer; for the solution, which corresponds $\alpha = \lambda =$ to 0, liquid head on boundary is constant in the course of entire process, the amount of liquid in layer grow/rises. With $-1/3 < \alpha < 0$, i.e., $1/2 < \lambda < 0$, the pressure head on boundary at the moment is infinite and decreases in the course of time to zero; amount of liquid,

initially equal as in all preceding cases, to zero, it increases in the course of time. With $\alpha = -1/3$, i.e., $\lambda = -1/2$, pressure head on boundary at the moment is infinite and in the course of time decreases to zero; the total amount of liquid in layer constantly during entire process - the liquid through boundary of $x = 0$ into layer does not enter. In all indicated cases on the boundary of layer $x = 0$ at any point in time, is reached the maximum for this torque/moment value of pressure head. With $-1/2 < \alpha < -1/3$, i.e., $-1 < \lambda < -1/2$, the liquid head on boundary at the moment is infinite and in the course of time decreases to zero. The total amount of liquid at the moment infinitely greatly and in the course of time decreases, strive for zero, so that on the boundary of layer liquid no longer flows into layer as in the preceding/previous cases, but it escape/ensues of layer. Then on the boundary of layer liquid head no longer is maximum; the maximum value of pressure head is reached in certain internal point of layer, different for different points in time.

6. Linearly increases with time the liquid head on the boundary of layer. Let us examine now the special case, which corresponds to the linear increase of liquid head on the boundary of layer, i.e., when $\alpha = 1$. Here

$$h(0,\ t)=\sigma(t-t_0);\quad \xi=\frac{x\sqrt{2}}{\sqrt{a\sigma(t-t_0)}} \tag{IV.1.45}$$

and equation (IV.1.7) assumes the form:

$$\frac{d^2f^2}{d\xi^2}+\frac{1}{2}\xi\frac{df}{d\xi}-\frac{1}{2}f=0. \tag{IV.1.46}$$

As it is not difficult to check, function

$$f(\xi,\ 1)=\begin{cases}1-\xi/2 & 0\leqslant\xi\leqslant\xi_0=2;\\ 0 & \xi_0\leqslant\xi\end{cases} \tag{IV.1.47}$$

satisfies an equation (IV.1.46) and all conditions of task; hence it is obtained

$$h(x, t) = \sigma(t - t_0) - \frac{x}{\sqrt{2a/\sigma}}; \quad 0 \leqslant x \leqslant \sqrt{2a\sigma}(t - t_0);$$
$$h(x, t) = 0; \quad \sqrt{2a\sigma}(t - t_0) \leqslant x < \infty. \qquad \text{(IV.1.48)}$$

. Page 73.

The coordinate of the leading edge of liquid x_0 (t) is expressed as follows:

$$x_0(t) = \sqrt{2a\sigma}(t - t_0), \qquad \text{(IV.1.49)}$$

a the constant velocity of propagation of leading edge

$$v_0 = \sqrt{2a\sigma}. \qquad (IV.1.50)$$

Thus, the curve/graph of the distribution of liquid head in layer is represented to those intercept/detached by the coordinate axes by the cut of the straight line, which is moved by in parallel to itself at constant velocity.

This qualitative result was experimentally checked by V. M.

Shestakov [122] on the slotted tray/chute, frequently used for the simulation of the free-flow motions of liquid in the porous medium (with the theory of slotted tray/chute it is possible to be acquainted according to the book of V. I. Aravin and S. N. Numerova [2]). Slotted tray/chute is two closely placed vertical glass plates; the slot between these plates has the impenetrable horizontal bottom and is connected with container sufficient large volume. The motion of viscous fluid in slot is subordinated to these regularities such as the free-flow motion of liquid in the porous medium; the point of connection of slot with container corresponds to the boundary of layer. Evenly heaving the level of glycerin in container, V. M. Shestakov [122] obtained the distribution of the levels of glycerin in tray/chute, which was well agreeing itself with the given theoretical result.

§2. Flat free-flow motions with the zero initial pressure head: maximum self-similar motions, axisymmetric self-similar motions.

1. Maximum self-simulation motions. Let us examine now for the same semi-infinite layer somewhat a different task. We will investigate motion in semi-infinite time interval $(-\infty, t)$; therefore the initial distribution of pressure head on layer unessentially.

Let us compose the complete copy of the arguments on which depends this solution. Besides coordinate x and time t, into this copy will enter also values h_0, $\varkappa$ and a. Then the dimensionality of all determining parameters of solution are represented in the form:

$$[x]=L;\ [t]=T;\ [a]=[h]^{-1}L^2T^{-1};\ [h_0]=[h];\ [\varkappa]=T^{-1}, \quad \text{(IV.2.4)}$$

where as before symbols L, T and [h] they mean with respect to the dimensionality of length, time and pressure head. Of five arguments (IV.2.4) with three independent dimensionality it is possible to compose two independent dimensionless combinations which are conveniently taken in the form:

$$x\sqrt{\frac{\varkappa}{ah_0}},\ \varkappa t;$$

hence on the basis of P-theorem the solution of the problem in question will be

$$h=h_0\varphi\left(\frac{x}{\sqrt{ah_0/\varkappa}},\ \varkappa t\right), \quad \text{(IV.2.5)}$$

where φ - dimensionless function.

Let us place now $t = t' + \tau$, where τ is an arbitrary constant. In this case, condition (IV.2.1) and equation (IV.2.3) as it is not difficult to check, they are record/written through the new variable t', just as through the previous variable, but condition (IV.2.2) assumes the form:

$$h(0, t') = h_0^* e^{\varkappa t'}; \quad h_0^* = h_0 e^{\varkappa \tau}. \qquad (IV.2.6)$$

Thus, shift/shear in time affects only certain transformation of value h_0, and the formulation of the problem turns out to be invariant with respect to the transformation group of transfer on

Let us compose the complete copy of the arguments on which depends this solution. Besides coordinate x and time t, into this copy will enter also values h_0, $\varkappa$ and a. Then the dimensionality of all determining parameters of solution are represented in the form:

$$[x]=L;\ [t]=T;\ [a]=[h]^{-1}L^2T^{-1};\ [h_0]=[h];\ [\varkappa]=T^{-1}, \qquad \text{(IV.2.4)}$$

where as before symbols L, T and [h] they mean with respect to the dimensionality of length, time and pressure head. Of five arguments (IV.2.4) with three independent dimensionality it is possible to compose two independent dimensionless combinations which are conveniently taken in the form:

$$x\sqrt{\frac{\varkappa}{ah_0}},\ \varkappa t;$$

hence on the basis of P-theorem the solution of the problem in question will be

$$h=h_0\varphi\left(\frac{x}{\sqrt{ah_0/\varkappa}},\ \varkappa t\right), \qquad \text{(IV.2.5)}$$

time; for determining of h in alternating/variable x, t', a, ϰ, h'$_0$ is obtained the same task, as for determining of h in variables (IV.2.4). That means on the basis of relaticnship/ratios (IV.2.5) and (IV.2.6) we have

$$h = h_0\varphi\left(x\sqrt{\frac{\varkappa}{ah_0}},\ \varkappa t\right) = h_0'\varphi\left(x\sqrt{\frac{\varkappa}{ah_0'}},\ \varkappa t'\right) = \\ = e^{\varkappa\tau}h_0\varphi\left(x\sqrt{\frac{\varkappa}{ah_0 e^{\varkappa\tau}}},\ \varkappa t - \varkappa\tau\right). \qquad \text{(IV.2.7)}$$

Hence it follows that with any τ occurs the identity

$$\varphi\left(x\sqrt{\frac{\varkappa}{ah_0}},\ \varkappa t\right) \equiv e^{\varkappa\tau}\varphi\left(x\sqrt{\frac{\varkappa}{ah_0 e^{\varkappa\tau}}},\ \varkappa t - \varkappa\tau\right). \qquad \text{(IV.2.8)}$$

Let us place now τ = t and obtain

$$\varphi\left(x\sqrt{\frac{\varkappa}{ah_0}},\ \varkappa t\right)=e^{\varkappa t}\varphi\left(x\sqrt{\frac{\varkappa}{ah_0e^{\varkappa t}}},\ 0\right)=e^{\varkappa t}f\left(x\sqrt{\frac{\varkappa}{ah_0e^{\varkappa t}}}\right). \qquad \text{(IV.2.9)}$$

Page 75.

Thus, function h, which depends from five arguments (IV.2.4), is

represented through the function of one argument:

$$h = h_0 e^{\varkappa t} f(\xi); \quad \xi = \frac{x e^{-1/_2 \varkappa t} \sqrt{\varkappa}}{\sqrt{a h_0}}. \tag{IV.2.10}$$

Substituting (IV.2.10) into the fundamental equation (IV.2.3), we obtain for function f (ξ) ordinary differential equation

$$\frac{d^2 f^2}{d\xi^2} + \frac{1}{2} \xi \frac{df}{d\xi} - f = 0. \tag{IV.2.11}$$

Substituting expression (IV.2.10) in condition at infinity (IV.2.1) and boundary condition (IV.2.2), we have boundary conditions for function f (ξ):

$$f(0) = 1; \quad f(\infty) = 0. \qquad \text{(IV.2.12)}$$

On the strength of the continuity of the liquid head and fluid flow, function f (ξ) must as before be continuous and have continuous derivative of square $df^2/d\xi$. We obtained, was concealed by form, for determining function f (ξ) a boundary-value problem of the same type, as boundary-value problems for the self-similar solutions, examined in the preceding/previous paragraph, and that corresponding to the value of the parameter α, equal to infinity, i.e., λ = 1. The effective calculation of function f (ξ) is fulfilled by the method, indicated in p. 4 of preceding/previous paragraph; the results of calculations were given in Table IV.1 and in Fig. IV.4. Function f (ξ) = f (ξ, 1) is identically equal to zero with $\xi > \xi_0 = 1.810$; leading edge x_0 (t) is moved, thus, according to the law

$$x_0(t) = 1{,}810 \sqrt{\frac{a h_0 e^{\nu t}}{\varkappa}}, \qquad \text{(IV.2.13)}$$

a the speed of its displacement/movement is equal to

$$v_0(t) = 0{,}905\sqrt{a\varkappa h_0 e^{\varkappa t}}. \qquad \text{(IV.2.14)}$$

The obtained solution is in a sense the limiting one for the self-similar solutions, examined in the preceding/previous paragraph. In fact, let us rely in formula (IV.1.6)

$$\sigma = h_0(\alpha\tau)^{-\alpha}, \qquad \text{(IV.2.15)}$$

where h_0 - certain constant of the dimensionality of pressure head; τ is a constant of the dimensionality of time, and, it is obvious,

these constants are selected with an accuracy to certain constant factor. Solution (IV.1.6) accepts in this case form

$$h=h_0\left(\frac{t-t_0}{\alpha\tau}\right)^{\alpha} f\left(\frac{x}{\sqrt{\frac{ah_0\tau}{\alpha^{\alpha}(1+\alpha)}\left(\frac{t-t_0}{\tau}\right)^{\alpha+1}}},\ \lambda\right). \qquad (IV.2.16)$$

. Page 76.

We will unlimitedly increase in this solution α in the initial torque/moment $t_0 \to -\infty$ according to the law

$$t_0 = -\alpha\tau. \qquad (IV.2.17)$$

. Revealing indeterminancy/uncertainty, we obtain, that with $\alpha \to \infty$

$$\left(\frac{t-t_0}{\alpha\tau}\right)^{\alpha} \rightarrow \exp\frac{t}{\tau};\ \ \frac{\alpha^{-\alpha}}{\alpha+1}\left(\frac{t-t_0}{\tau}\right)^{\alpha+1} \rightarrow \exp\frac{t}{\tau};\ \ \lambda \rightarrow 1. \qquad (IV.2.18)$$

Equation (IV.1.7) in the limit with $\alpha \rightarrow \infty$ transfer/converts to equation (IV.2.11), and conditions (IV.1.8) and (IV.1.9) coincide with conditions (IV.2.12); $f(\xi, \lambda) \rightarrow f(\xi, 1) = f(\xi)$.

Designating τ through the aaaaa, we obtain, that in α ∞ the solution (IV.2.16) approaches solution (IV.2.10). Therefore solution (IV.2.10) was called by maximum self-similar solution. This solution was obtained in G. I. Barenblatt's work [8]. Maximum self-similar solutions are of and fundamental interest in that relation, that for the proof of the self-similarity of these solutions already insufficient considerations of dimensional analysis, i.e., insufficient invariance of the formulation of the problem relative to the transformation group of the similarity of values with independent

dimensionality, as this was in previously examined self-similar problems, and it is required additionally to use the invariance of the formulation of the problem relative to one additional group - the transformation group of transfer in time.

Give in the examination of the maximum self-similar problem of reasoning bear common character and can be applied in many other tasks. It is obvious that the maximum self-similar motions exist always, if the system of the fundamental equations of the task in question has usual exponential type self-similar solutions with arbitrary exponent (which can take how conveniently great significance) and is invariant relative to the transformation of the transfer of the corresponding coordinate. As an example it is possible to indicate the task of boundary layer in the incompressible fluid, and also the task of the one-dimensional unsteady motions of gas. Obtained for these tasks self-similar solutions, which contain exponential functions of independent variables [136, 103], during the passage to the limit, analogous to that made in the task of the theory of filtration [1] in question, give the maximum self-similar solutions, obtained by Goldstein and Stanyukovich [137, 109] by means of formal setting.

FOOTNOTE 1. See G. I. Barenblatt's article [8] and L. I. Sedov's book [102]. ENDFOOTNOTE.

Task. On boundary of x = 0 semi-infinite layer with impenetrable horizontal confining stratum is assigned the flow (flow rate) of liquid as exponential function of time

$$-\frac{1}{2}C\left(\frac{\partial h^2}{\partial x}\right)_{x=0}=\tau(t-t_0)^{\beta};\ \ \beta>-1,\ \tau>0. \qquad \text{(IV.2.19)}$$

. The initial pressure head in all layer is equal to zero.

Page 77.

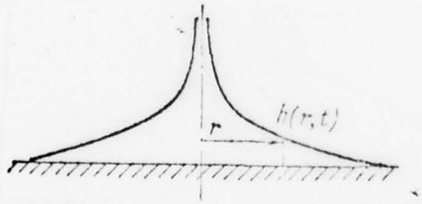


Fig. IV.6.

The solution of problem is represented in the form:

$$h=\left[\frac{6a\tau^{2}(t-t_{0})^{2\beta+1}}{C^{2}M^{2}(\lambda)(\beta+2)}\right]^{1/3} f\left\{x\left[\frac{2CM(\lambda)(\beta+2)^{2}}{9a^{2}\tau(t-t_{0})^{\beta+2}}\right]^{1/3}, \lambda\right\}, \qquad \text{(IV.2.20)}$$

where $M(\lambda) = df^{2}(0, \lambda)/d\xi$ (see Fig. IV.5 and Table IV.2), but the coordinate of the leading edge of liquid $x_0(t)$ - in the form:

$$x_{0}(t)=\xi_{0}(\lambda)\left[\frac{9a^{2}\tau(t-t_{0})^{\beta+2}}{2CM(\lambda)(\beta+2)^{2}}\right]^{1/3}. \qquad \text{(IV.2.21)}$$

2. Axisymmetric self-similar motions. During the axisymmetrical flat free-flow motions of liquid, the liquid head h satisfies an equation

$$\frac{\partial h}{\partial t}=a\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h^2}{\partial r}\right);\quad a=\frac{C}{2m}=\frac{k\rho g}{2\mu m},\qquad \text{(IV.2.22)}$$

where r - the distance of the point of layer in question from the axis of symmetry.

Let us examine the following task. Let (Fig. IV.6) into the infinite layer, bounded below by impenetrable horizontal surface - confining stratum, through the hole whose radius is negligible, it begins the pumping of liquid. Let us assume that the initial liquid head in layer is equal to zero, so that the initial condition and condition at infinity take the form:

$$h(r, t_0)=0;\quad h(\infty, t)=0.\qquad \text{(IV.2.23)}$$

Let us assume further that the flow rate of the inject/begun

rocking liquid changes in the course of time according to power law. Expression for the complete fluid flow rate, pumped through the hole with a radius of R, takes the form:

$$q(t)=2\pi R\left(-Ch\frac{\partial h}{\partial r}\right)_{r=R}=-\pi C\left(r\frac{\partial h^2}{\partial r}\right)_{r=R}. \qquad \text{(IV.2.24)}$$

• By hypothesis, a radius of the hole is negligible (below we let us pause at reasons, on which this assumption it is possible to make for the majority of real motions); therefore, we can say that R = 0; since fluid flow rate, inject/begun rocking into hole, it is changed according to power law, boundary condition on hole assumes the form:

$$-\pi C\left(r\frac{\partial h^2}{\partial r}\right)_{r=0}=\tau(t-t_0)^\beta, \qquad \text{(IV.2.25)}$$

where $\tau > 0$ and $\beta > -1$. Specifically, the case $\beta = 0$ corresponds to the pumping of liquid into layer with constant flow rate/consumption. Thus, the solution of problem satisfies an equation (IV.2.22) and conditions (IV.2.23) and (IV.2.25).

Page 78.

As before, utilizing P-theorem of dimensional analysis, it is possible to show that this solution is self-similar and is represented in the form:

$$h=\left[\frac{\tau}{\pi C}(t-t_0)^{\beta}\right]^{1/2} f_1(\xi, \lambda). \qquad \text{(IV.2.26)}$$

. Here

$$\xi = r\sqrt[4]{\frac{4a^2\tau(t-t_0)^{\beta+2}}{\pi C(\beta+2)^2}}; \quad \lambda=\frac{\beta}{\beta+2} \qquad \text{(IV.2.27)}$$

are two independent dimensionless combinations of the determining parameters of solution; other independent combinations of these parameters do not exist. Constant factor is again introduced into formula for ξ for the target/purpose of convenience in the subsequent presentation. As m before, the unknown function must be continuous and have continuous derivative of square. Substituting expression (IV.2.26) in equation (IV.2.22) and conditions (IV.2.23) and (IV.2.25), we find that function f_1 (ξ, λ) it satisfies an equation

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df_1}{d\xi}\right)+\frac{1}{2}\xi\frac{df_1}{d\xi}-\lambda f_1=0 \qquad \text{(IV.2.28)}$$

under the conditions

$$\xi\frac{df_1}{d\xi}\Big|_{\xi=0}=-1; \quad f_1(\infty, \lambda)=0. \qquad \text{(IV.2.29)}$$

The investigation of this boundary-value problem is carried out to analogously preceding/previous; also by only form is constructed function f_1 (ξ, λ), that differs from zero only with $0 \leqslant \xi \leqslant \xi_1(\lambda)$, where ξ_1 (λ) is certain function ξ, and with $\xi \geqslant \xi_1$ (λ) identically equal to zero. Function f_1 (ξ, λ) with $\xi \to 0$ has a special feature/peculiarity as not difficult to see from the first condition (IV.2.29):

$$f_1(\xi, \lambda) \approx \sqrt{\ln \frac{1}{\xi}} \quad (\xi \to 0). \qquad \text{(IV.2.30)}$$

• The second condition (IV.2.29) can be given to another form: multiplying equation (IV.2.28) by ξ and integrating within limits from $\xi = 0$ to $\xi = \infty$, we obtain, utilizing both conditions (IV.2.29) and conditions

$$\left(\xi \frac{df_1^0}{d\xi}\right)_{\xi=\infty} = 0; \quad [\xi f_1(\xi, \lambda)]_{\xi=0} = 0, \qquad \text{(IV.2.31)}$$

the following integral relationship/ratio:

$$\int_0^\infty \xi f_1(\xi, \lambda)\, d\xi = \int_0^{\xi_1(\lambda)} \xi f_1(\xi, \lambda)\, d\xi = \frac{1}{1+\lambda}. \qquad \text{(IV.2.32)}$$

. The first condition (IV.2.31) directly follows from the condition which satisfies function f_1 (ξ, λ) at infinity, since if limit ξ $df^{?}_1/d\xi$ with $\xi \rightarrow \infty$ was not to equal to zero, then is function f_1 (ξ, λ) it would not approach zero with $\xi \rightarrow \infty$. The second condition (IV.2.31) directly follows from (IV.2.30).

Page 79.

The effective calculation of function f_1 (ξ, λ) is conveniently carried out as follows. We construct the solution of the problem of Cauchy Φ_1 (ξ, λ) for equation (IV.2.28), that becomes zero with ξ = 1 and having in this point final first-order derivative.

Investigation, in accuracy analogous that which was given in p. 3 §1, shows that this derivative is equal to -1/4. To construct the solution of the problem of Cauchy is convenient as follows: near ξ = 1 it is possible to present solution in the form of the series with the aid of which is located the proper number of initial values, whereupon is applied the method of the numerical integration of adamsa - Stormer. Further numerically is calculated value

$$\lim_{\xi\to 0}\left(\xi\frac{d\Phi_1^2}{d\xi}\right) = -N(\lambda).$$

Value N (λ) is not equal to unity; therefore function, equal & Φ_1 (ξ, λ) with ξ < 1 and identically equal to zero with $\xi \geqslant 1$, satisfies all conditions of boundary-value problem (IV.2.28) - (IV.2.29), except the first condition (IV.2.29). We will use the now fact that as it is not difficult to show, equation (IV.2.28) and the second boundary condition (IV.2.29) are invariant relative to transformation group:

$$\Phi_2(\xi, \lambda) = \frac{1}{\mu^2}\Phi_1(\xi\mu, \lambda), \qquad \text{(IV.2.33)}$$

therefore with arbitrary positive μ the function of Φ_2 (ξ, λ) satisfies an equation (IV.2.28) and the second boundary condition (IV.2.29). but

$$\left(\xi\frac{d\Phi_2^2}{d\xi}\right)_{\xi=0} = \frac{1}{\mu^4}\left(\xi\frac{d\Phi_1^2(\xi, \lambda)}{d\xi}\right)_{\xi=0} = -\mu^{-4}N(\lambda). \qquad \text{(IV.2.34)}$$

After selecting $\mu = \mu_* = \sqrt[4]{N(\lambda)}$ so that $\left(\xi\frac{d\Phi_2^2}{d\xi}\right)_{\xi=0} = -1,$ we will obtain that the function

$$f_1(\xi, \lambda) = \begin{cases} \frac{1}{\sqrt{N(\lambda)}}\Phi_1(\xi\sqrt[4]{N(\lambda)};\lambda); & [0 \leqslant \xi \leqslant \xi_1(\lambda)] = [N(\lambda)]^{-1/4}; \\ 0 & [\xi \geqslant \xi_1(\lambda)] \end{cases} \qquad \text{(IV.2.35)}$$

satisfies all conditions of boundary-value problem (IV.2.28) - (IV.2.29).

Table IV.3 depicts those obtained as a result of carried out thus the calculations of the value of function f_1 (ξ, λ) for λ within limits from zero to one through 0.1.

Page 80.

Table IV-3.

	λ = 0,00	λ = 0,05	λ = 0,10	λ = 0,15.	λ = 0,20	λ = 0,25
ξ	0,01119	0,01096	0,01075	0,01056	0,01037	0,01020
f_1	2,534	2,525	0,517	2,510	2 502	2,497
ξ	0,01758	0,01722	0,01689	0,01659	0,01630	0,01603
f_1	2,412	2,402	2,394	2,387	2,379	2,373
ξ	0,02557	0,02505	0,02457	0,02413	0,02371	0,02331
f_1	2,307	2,296	2,287	2,280	2,272	2,265
ξ	0,03836	0,03758	0,03686	0,03620	0,03556	0,03497
f_1	2,186	2,176	2,166	2,158	2,149	2,142
ξ	0,05754	0,05637	0,05529	0,05430	0,05334	0,05246
f_1	2,059	2,047	2,037	2,029	2,020	2,012
ξ	0,08950	0,08769	0,08601	0,08446	0,08298	0,08160
f_1	1,911	0,898	1,887	1,878	1,868	1,860
ξ	0,1279	0,1252	0,1229	0,1207	0,1185	0,1166
f_1	1,782	1,769	1,757	1,747	1,737	1,728
ξ	0,1918	0,1879	0,1843	1,1810	0,1778	0,1749
f_1	1,624	1,609	1,597	1,586	1,574	1,565
ξ	0,2685	0,2631	0,2580	0,2534	0,2489	0,2448
f_1	1,480	1,464	1,451	1,439	1,427	1,417
ξ	0,3580	0,3507	0,3440	0,3378	0,3319	0,3264
f_1	1,345	1,329	1,315	1,302	1,289	1,278
ξ	0,4731	0,4635	0,4546	0,4464	0,4386	0,4313
f_1	1,202	1,185	1,170	1,156	1,143	1,132
ξ	0,6137	0,6013	0,5898	0,5792	0,5216	0,5596
f_1	1,054	1,036	1,021	1,006	1,045	0,9809
ξ	0,7416	0,7266	0,7127	0,6998	0 6638	0,6761
f_1	0,9347	0,9170	0,9012	0,8868	0,8962	0,8610
ξ	0,8950	0,8769	0,8601	0,8446	0,7824	0,8160
f_1	0,8042	0,7868	0,7713	0,7571	0,7855	0,7318
ξ	1,048	1,027	1,008	0,9894	0,9246	0,9559
f_1	0,6830	0,6665	0,6517	0,6381	0,6639	0,6139
ξ	1,227	1,203	1,180	1,158	1,043	1,119
f_1	0,5494	0,5344	0,5211	0,5087	0,5693	0,4867
ξ	1,381	1,353	1,327	1,303	1,185	1,259
f_1	0,4394	0,4262	0,4146	0,4037	0,4621	0,3845
ξ	1,509	1,478	1,450	1,424	1,304	1,376
f_1	0,3498	0,3386	0,3288	0,3193	0,3768	0,3029
ξ	1,637	1,603	1,573	1,544	1,422	1,492
f_1	0,2616	0,2527	0,2450	0,2373	0,2946	0,2242
ξ	1,739	1,704	—	1,641	1,541	—
f_1	0,1917	0,1848	—	0,1730	0,2147	—
ξ	1,841	1,804	1,769	1,737	1,660	1,679
f_1	0,1221	0,1175	0,1134	0,1096	0,1368	0,1029
ξ	1,943	1,904	1,868	1,834	1,802	1,772
f_1	0,05233	0,05030	0,04845	0,4677	0,04520	0,04377
ξ_0	2,046	2,004	1,966	1,931	1,897	1,865

	λ = 0,30	λ = 0,35	λ = 0,40	λ = 0,45	λ = 0,50
ξ	0,01004	0,009885	0,009740	0,009603	0,009472
f_1	2,491	2,486	2,481	2,477	2,472
ξ	0,01577	0,01563	0,01531	0,01509	0,01488

Page 81.

Table IV-3 (continuation).

	$\lambda=0,30$	$\lambda=0,35$	$\lambda=0,40$	$\lambda=0,45$	$\lambda=0,50$
f_1	2,367	2,361	2,356	2,351	2,347
ξ	0,02294	0,02259	0,02226	0,2195	0,02165
f_1	2,259	2,253	2 248	2,243	2,238
ξ	0,03442	0,03389	0,03340	0,3292	0,03247
f_1	2,136	2,130	2,124	2,119	2,114
ξ	0,05163	0,05084	0,05009	0,04939	0,04871
f_1	2,005	1,999	1,992	1,987	1,982
ξ	0,08031	0,07908	0,07792	0,7682	0,7577
f_1	1,852	1,845	1,839	1,833	1,827
ξ	0,1147	0,1130	0,1113	0,1097	0,1082
f_1	1,720	1,712	1,705	1,699	1,692
ξ	0,1721	0,1808	0,1670	0,1646	0,1624
f_1	1,556	1,520	1,540	1,532	1,526
ξ	0,2409	0,2372	0,2332	0,2305	0,2273
f_1	1,407	1,398	1,390	1,382	1,375
ξ	0,3212	0,3163	0,3117	0,3073	0,3031
f_1	1,268	1,259	1,250	1,242	1,234
ξ	0,4245	0,4180	0,4119	0,4061	0,4005
f_1	0,1121	1,111	1,102	1,093	1,085
ξ	0,5507	0,5423	0,5343	0,5268	0,5196
f_1	0,9698	0,9695	0,9498	0,9409	0,9325
ξ	0,6654	0,6553	0,6457	0,6585	0,6278
f_1	0,8498	0,8394	0,8297	0,7982	0,8122
ξ	0 8031	0,7908	0,7792	0,7682	0,7577
f_1	0,7208	0,7105	0,7009	0,6920	0,6837
ξ	0,9407	0,9264	0,9128	0,8999	0,8876
f_1	0,6033	0,5935	0,5843	0,5758	0,5678
ξ	1,101	1,085	1,069	1,054	1,039
f_1	0,4771	0,4682	0,4599	0,4521	0,4448
ξ	1,239	1,220	1,202	1,185	1,169
f_1	0,3760	0,3682	0,3608	0,3540	0 3476
ξ	1,330	1,311	1,291	1,273	1,256
f_1	0,3115	0,3046	0,2981	0,2920	0,2863
ξ	1,468	1,446	1,425	1,404	1,386
f_1	0,2184	0,2131	0,2081	0,2034	0,1990
ξ	1,560	1,536	1,514	1,493	1,472
f_1	0,1585	0,1543	0,1505	0,1469	0,1435
ξ	1,652	1,627	1,603	1,580	1,559
f_1	0,09992	0,09716	0,09459	0,09219	0,08993
ξ	1,744	1,717	1,692	1,668	1,645
f_1	0 04245	0,04121	0,04006	0,03899	0,03798
ξ_1	1,836	1,808	1,781	1,756	1,732
	$\lambda=0,55$	$\lambda=0,60$	$\lambda=0,65$	$\lambda=0,70$	$\lambda=0,75$
ξ	0,009347	0,009232	0,009114	0,009000	0,008901
f_1	2,468	2,466	2,463	2,460	2,455
ξ	0,01469	0,01451	0,01432	0,01416	0,01399
f_1	2,343	2,341	2,337	2,334	2,329
ξ	0,02137	0,02110	0,02083	0,02059	0,02034
f_1	2,233	2,232	2,227	2,224	2,219

Page 82.

Table IV-3 (continuation).

	$\lambda=0,55$	$\lambda=0,60$	$\lambda=0,65$	$\lambda=0,70$	$\lambda=0,75$
ξ	0,03205	0,03165	0,03125	0,03089	0,03052
f_1	2,109	2,107	2,102	2,099	2,093
ξ	0,04807	0,04748	0,04687	0,04633	0,04577
f_1	1,976	1,974	1,969	1,965	1,960
ξ	0,07478	0,07386	0,07291	0,07208	0,07120
f_1	1,822	1,818	1,813	1,809	1,803
ξ	1,1068	0,1055	0,1042	0,1030	0,1017
f_1	0,687	1,683	1,678	1,673	1,667
ξ	1,1602	0,1583	0,1562	0,1544	0,1526
f_1	1,520	1,516	1,509	1,505	1,499
ξ	0,2243	0,2216	0,2187	0,8162	0,2136
f_1	1,368	1,363	1,357	1,352	1,345
ξ	0,2291	0,2954	0,2917	0,2883	0,2849
f_1	1,227	1,221	1,215	1,209	1,202
ξ	0,3953	0,3904	0,3854	0,3810	0,3764
f_1	1,077	1,071	1,065	1,058	1,052
ξ	0,5128	0,5065	0,5000	0,4942	0,4883
f_1	0,9246	0,9181	0,9111	0,9047	0,8977
ξ	0,6196	0,6120	0,6041	0,5972	0,5900
f_1	0,8042	0,7975	0,7905	0,7839	0,7769
ξ	0,7478	0,7386	0,7291	0,7208	0,7120
f_1	0,6758	0,6691	0,6621	0,6556	0,6488
ξ	0,8760	0,8652	0,8541	0,8443	0,8341
f_1	0,5603	0,5538	0,5472	0,5409	0,5345
ξ	1,026	1,013	1,000	0,9885	0,9765
f_1	0,4380	0,4320	0,4259	0,4202	0,4144
ξ	1,154	1,140	1,125	1,112	1,099
f_1	0,3416	0,3362	0,3309	0,3260	0,3209
ξ	1,239	1,224	1,208	1,194	1,180
f_1	0,2810	0,2762	0,2715	0,2671	0,2626
ξ	1,368	1,351	1,333	1,277	1,261
f_1	0,1949	0,1912	0,1875	0,2111	0,2073
ξ	1,453	1,435	1,417	1,359	1,343
f_1	0,1403	0,1375	0,1346	0,1578	0,1547
ξ	1,538	1,519	1,500	1,442	1,424
f_1	0,08782	0,08591	0,08396	1,099	0,1047
ξ	1,624	1,604	1,583	1,565	1,546
f_1	0,03703	0,03617	0,03529	0,03453	0,03374
ξ_1	1,709	1,688	1,667	1,647	1,628

	$\lambda=0,80$	$\lambda=0,85$	$\lambda=0,90$	$\lambda=0,95$	$\lambda=1,00$
ξ	0,008800	0,008707	0,008610	0,08522	0,008434
f_1	2,453	2,452	2,448	2,447	2,444
ξ	0,01383	0,01368	0,01230	0,01096	0,01084
f_1	2,327	2,326	2,319	2,318	2,314
ξ	0,02011	0,01990	0,01968	0,01948	0,01928
f_1	2,216	2,216	2,211	2,210	2,206
ξ	0,03017	0,02985	0,02952	0,02922	0,02892
f_1	2,091	2,090	2,085	2,083	2,080
ξ	0,04526	0,04478	0,04428	0,04383	0,04338

Page 83.

Table IV-3 (continuation).

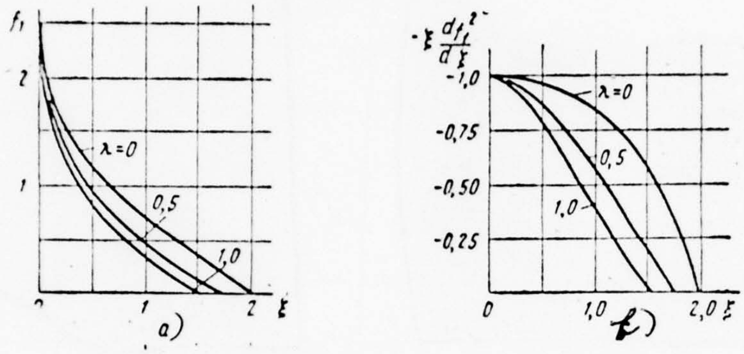
	$\lambda = 0,80$	$\lambda = 0,85$	$\lambda = 0,90$	$\lambda = 0,95$	$\lambda = 1,00$
f_1	1,957	1,955	1,950	1,948	1,945
ξ	0,07040	0,06965	0,06888	0,6818	0,06747
f_1	1,800	1,798	1,793	1,791	1,787
ξ	0,1006	0,09951	0,09840	0,09740	0,09639
f_1	1,664	1,661	1,656	1,654	1,649
ξ	0,1509	0,1493	0,1476	0,1461	0,1446
f_1	1,495	1,492	1,486	1,483	1,479
ξ	0,2112	0,2090	0,2066	0,2045	0,2024
f_1	1,341	1,338	1,332	1,329	1,323
ξ	0,2816	0,2786	0,2755	0,2727	0,2699
f_1	1,198	0,194	1,188	1,184	1,179
ξ	0,3721	0,3682	0,3641	0,3604	0,3470
f_1	1,046	1,042	1,036	1,032	1,043
ξ	0,4827	0,4777	0,4723	0,4675	0,4627
f_1	0,8923	0,8876	0,8816	0,8772	0,8720
ξ	0,5833	0,5772	0,5707	0,5649	0,5591
f_1	0,7714	0,7666	0,7606	0,7560	0,7508
ξ	0,7040	0,6966	0,6888	0,6818	0,6747
f_1	0,6433	0,6384	0,6326	0,6280	0,6230
ξ	0,8046	0,7961	0,7872	0,7792	0,7711
f_1	0,5474	0,5426	0,5370	0,5325	0,5277
ξ	0,9253	0,9155	0,9053	0,8961	0,8868
f_1	0,4424	0,4379	0,4328	0,4285	0,4240
ξ	1,086	1,075	1,023	0,9740	1,002
f_1	0,3166	0,3127	0,3381	0,3647	0,3302
ξ	1,207	1,154	1,181	1,052	1,080
f_1	0,2311	0,2553	0,2242	0,3047	0,2724
ξ	1,287	1,234	1,260	1,130	1,195
f_1	0,1777	0,2010	0,1720	0,2482	0,1920
ξ	1,368	1,314	1,338	1,247	1,272
f_1	0,1271	0,1496	0,1226	0,1694	0,1424
ξ	1,408	1,393	1,417	1,364	1,388
f_1	0,1027	0,1009	0,07594	0,09730	0,07326
ξ	1,529	1,513	1,496	1,481	1,465
f_1	0,03302	0,3237	0,03169	0,03109	0,03048
ξ_1	1,609	1,592	1,575	1,558	1,542

For convenience in the calculation of fluid flow rate according to Fig. IV.7a and b are given the corresponding values f_1 (ξ) and $\left(-\xi \frac{df_1^2}{d\xi}\right)_{\xi=0}$ with $\lambda = 0$ (an abrupt change in the flow rate). On Fig. IV.8 is constructed the plotted function ξ_1 (λ).

Coordinate r_0 the driving leading edge of liquid is expressed by relationship/ratio

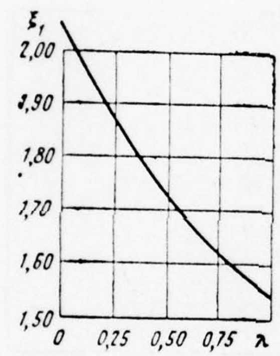
$$r_0 = \xi_1(\lambda) \sqrt[4]{\frac{4a^2\tau (t-t_0)^{\beta+2}}{\pi C (\beta+2)^2}} = \sqrt[4]{\frac{C\tau (t-t_0)^{\beta+2}}{\pi m^2 (\beta+2)^2 N(\lambda)}}. \quad \text{(IV.2.36)}$$

Fig. IV.7.



[redacted]

Fig IV.8.



Specifically, in the case of constant flow rate/consumption of the inject/begun rocking liquid q (t) = τ, i.e., with β = 0, expression for a liquid head is represented in the form:

$$h=\left(\frac{\tau}{\pi C}\right)^{1/4} f_1\left(\frac{r}{\left(\frac{C\tau}{4\pi m^2}\right)^{1/4}\sqrt{t-t_0}};\ 0\right). \qquad \text{(IV.2.37)}$$

The coordinate of the leading edge of liquid in this case is expressed as

$$r_0(t)=1{,}537\left(\frac{a^2\tau}{C}\right)^{1/4}\times$$
$$\times\sqrt{t-t_0}=1{,}087\left(\frac{C\tau}{m^2}\right)^{1/4}\sqrt{t-t_0}. \qquad \text{(IV.2.38)}$$

§3. the self-similar motions of liquid and gas by plane waves in semi-infinite layer with nonzero the initial pressure of gas or liquid level.

1. Self-similar flat free-flow motions on the nonzero initial level of liquid. Let us examine again the flat free-flow motions of the incompressible fluid in the semi-infinite layer, bounded below by

horizontal confining stratum, and from the side - by the vertical flat/plane boundary, along another side of which is arrange/located the reservoir, filled by liquid. Let us assume that the initial level of liquid in the layer above the confining stratum is constant and equal to certain value h_0, different from zero (case $h_0 = 0$ was examined above).

Page 85.

Let us suppose further that at the moment liquid level in reservoir suddenly changes, it reaches certain value h_1, greater or less h_0 (but first not equal to zero; case $h_1 = 0$ will be examined especially) and then it remains constant. It is obvious that increase h_1 the free surface of liquid depends only on time t and coordinate x, calculated along the normal to flat/plane boundary; on same boundary we will give the value of coordinate x, equal to zero, so that equation for h it takes the form:

$$\frac{\partial h}{\partial t} = a\frac{\partial^2 h^2}{\partial x^2}; \quad a = \frac{k\rho g}{2\mu m}. \qquad \text{(IV.3.1)}$$

On the strength of the constancy of the initial level of liquid, the initial condition and condition at infinity are represented in the form:

$$h(x, 0) = h_0; \quad h(\infty, t) = h_0, \qquad \text{(IV.3.2)}$$

a boundary conditions of layer x = 0 it takes the form

$$h(0, t) = h_1. \qquad \text{(IV.3.3)}$$

Thus, increase h of floating surface depends on the following values:

$$x, t, a, h_0, h_1, \qquad \text{(IV.3.4)}$$

having dimensionality

$$[x]=L;\ [t]=T;\ [a]=[h]^{-1}L^2T^{-1};\ [h_0]=[h_1]=[h]$$

(L is dimensionality of length, Vol. - the dimensionality of time, [h] - the dimensionality of pressure head, which we right to accept independent of dimensionality length). Of values (IV.3.4) it is possible, obviously, to compose two independent dimensionless combinations as which convenient to select

$$\xi=\frac{x}{\sqrt{ah_1t}},\quad \lambda=\frac{h_0}{h_1}, \qquad \text{(IV.3.5)}$$

so that the motion in question turns out to be self-similar and function h is represented in the form:

$$h=h_1F(\xi,\lambda). \qquad \text{(IV.3.6)}$$

Substituting this representation of function h in equation (IV.3.1) and conditions (IV.3.2) and (IV.3.3), we obtain for determining function F (ξ, λ) equation

$$\frac{d^2F^2}{d\xi^2}+\frac{1}{2}\xi\frac{dF}{d\xi}=0 \qquad \text{(IV.3.7)}$$

and boundary conditions

$$F(0, \lambda)=1, \quad F(\infty, \lambda)=\lambda. \qquad \text{(IV.3.8)}$$

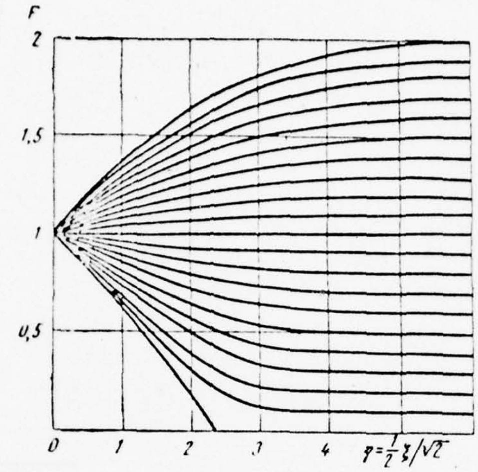
In this case not one of the boundary conditions (IV.3.8) no

longer is invariant relative to transformation group

$$\Phi(\xi) = \frac{1}{\mu^2} F(\mu\xi, \lambda),$$

although the equation (IV.3.7) as before invariant relative to this group.

Fig. IV.9.



~~Fig. IV.10.~~

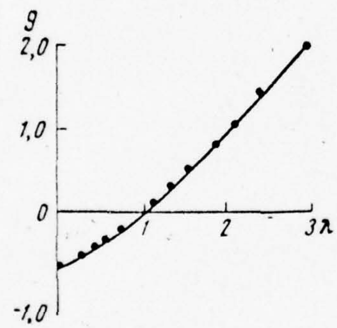


Fig IV.10

Due to this fact the determination of function F (ξ, λ) does not succeed in leading to the Cauchy problem, which strongly complicates its effective calculation. Therefore calculations were carried out in computer (led calculations N. P. Trifon).

The values of function F (ξ, λ) are represented for different λ (from 0 to 2 with space 0.1) on Fig. IV.9, here is depicted limiting case λ = as 0, corresponding h_0 = to 0 and examined above. Figure IV.10, gives the values of the function of $g(\lambda) = \frac{dF^2}{d\xi}\Big|_{\xi=0}$, which determine fluid flow through the boundary of layer q according to relationship/ratio

$$q = -\frac{k\rho g}{2\mu}\,\frac{h_0^2 g(\lambda)}{\sqrt{a_1 ht}}. \qquad \text{(IV.3.9)}$$

Let us present in equation (IV.3.7) term 1/2/ξ dF/dξ in the form of $\frac{1}{2}\xi\frac{d[F(\xi,\lambda)-\lambda]}{d\xi}$, let us multiply its both parts on ξ and integrate this equation from ξ = 0 to ξ = ∞; we will obtain 15. Page

87.

$$\int_0^\infty \xi \frac{d^2F^2}{d\xi^2} d\xi + \frac{1}{2}\int_0^\infty \xi^2 \frac{d}{d\xi}[F(\xi,\lambda)-\lambda]\,d\xi = \left[\xi\frac{dF^2}{d\xi}\right]_{\xi=0}^{\xi=\infty} - \\ -\int_0^\infty \frac{dF^2}{d\xi}d\xi + \frac{\xi^2}{2}[F(\xi,\lambda)-\lambda]_{\xi=0}^{\xi=\infty} - \int_0^\infty \xi[F(\xi,\lambda)-\lambda]\,d\xi. \tag{IV.3.10}$$

The study which we give here below shows that function F (ξ, λ) it approaches its extreme value with $\xi \rightarrow \infty$ very rapidly, on exponential law. Therefore, and also taking into account that with $\xi = 0$ F (ξ, λ) and $dF_2/d\xi$ are final, obtain

$$\left[\xi\frac{dF^2}{d\xi}\right]_{\xi=0}^{\xi=\infty} = \frac{\xi^2}{2}[F(\xi,\lambda)-\lambda]_{\xi=0}^{\xi=\infty} = 0.$$

Since, obviously,

$$\int_0^\infty \frac{dF^2}{d\xi}d\xi = F^2(\infty,\lambda) - F^2(0,\lambda),$$

the expression (IV.3.10) gives the integral relationship/ratio which

satisfies function F (ξ, λ):

$$\int_0^\infty \xi [F(\xi, \lambda) - \lambda] d\xi = 1 - \lambda^2. \qquad \text{(IV.3.11)}$$

. The self-similar solutions of the problem examined above of the flat free-flow filtration of liquid in nonzero initial level were found P. R. Polubarinovoy-Koch [92, 93]. To the existence of self-similar solutions of such type in the tasks of the unsteady filtration of liquid and gas, it was shown in the works of Boussinesq [133] and of Leybenzon [72]; however, their neither qualitative investigations nor numerical calculation in these works led it was not.

2. Filtration of liquid from semi-infinite layer into empty reservoir. Case $h_1 = 0$ needs separate examination. Here of the remaining determining parameters it is possible to compose only one dimensionless combination $\xi = x/\sqrt{ah_0 t}$, so that the increase of

floating surface h is represented in the form:

$$h=h_0\varphi(\xi). \qquad (IV.3.12)$$

Function h satisfies an equation as before of Boussinesq (IV.3.1), the initial condition and condition at infinity (IV.3.2), and also boundary condition

$$h(0,t)=0, \qquad (IV.3.13)$$

whence it is obtained, that the function ϕ (ξ) satisfies an equation

$$\frac{d^2\varphi^2}{d\xi^2}+\frac{1}{2}\xi\frac{d\varphi}{d\xi}=0 \qquad (IV.3.14)$$

under the conditions

$$\varphi(0)=0;\quad \varphi(\infty)=1. \qquad (IV.3.15)$$

In this case the equation (IV.3.14) and the first condition (IV.3.15) are invariant relative to transformation group $f(\xi)=\mu^{-2}\varphi$.

Page 88.

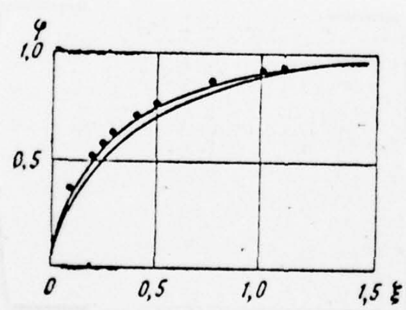


Fig. IV.11

Therefore, if ϕ (ξ) satisfies an equation (IV.3.14) and the first condition (IV.3.15), then also f (ξ) satisfies these relationship/ratios with any $\mu > 0$. This makes it possible to bring together the determination of function to the solution of the problem of Cauchy for equation (IV.3.14). However, in this case there is no even need for solving the problem of Cauchy in view of the fact that the problem in question turns out to be within the precision of the mathematical equivalent basis to the task of the boundary-layer theory - of boundary layer n to flat/plane plate.

In fact, let us rely in equation (IV.3.1) $z(x, t) = h_0^2 - h^2$. Then this equation is reduced to the following:

$$\frac{\partial z}{\partial t} = 2a\sqrt{h_0^2 - z^2}\,\frac{\partial^2 z}{\partial x^2}, \qquad \text{(IV.3.16)}$$

a condition (IV.3.2) and (IV.3.3) they will be rewritten as follows:

$$z(\infty, t) = 0; \quad z(0, t) = h_0^2;$$
$$z(x, 0) = 0. \qquad (IV.3.17)$$

But equation (IV.3.16) and condition (IV.3.17) coincide with the fundamental equation in the form of Mises and the conditions of the mentioned task of boundary-layer theory [59], if we replace t by the longitudinal coordinate x, x - by the function of the current of ψ, 2a - by the viscosity of the liquid of ν, h_0 - by velocity of incident flow U, whereupon z it expresses value $U^2 - u^2$, where u is the longitudinal velocity of flow. Thus, the increase of floating surface h corresponds to the longitudinal velocity of flow u in the task of boundary layer. Let us note now that we determine the dependence of function ϕ, equal to h/h_0, from alternating/variable $\xi = x/\sqrt{ah_0 t}$, which in the terms of boundary layer corresponds to the dependence of function u/U of alternating/variable $\psi\sqrt{\frac{2}{\nu U x}}$. As is known of boundary-layer theory,

$$\frac{u}{U} = \zeta'(\eta); \quad \frac{\psi}{\sqrt{\nu U x}} = \zeta(\eta), \qquad (IV.3.18)$$

where the $\zeta(\eta)$ - the tabulated function of Blasius, the table of the values of derivative of which has in each management/manual on hydrodynamics, and η are a dimensionless variable of Blasius, is equal $y\sqrt{\frac{U}{\nu x}}$ (y - transverse coordinate in boundary layer). Thus, we must find dependence ζ' on alternating/variable $\sqrt{2\zeta}$ and then, set/assuming $\zeta' = \varphi$, $\sqrt{2\zeta} = \xi$, we will obtain the unknown function Φ (ξ).

Page 89.

Figure IV.11 gives those which were obtained is concealed by the form of the value of function Φ (ξ).

Let us determine now fluid flow, which escape/ensues in

reservoir from layer. We have

$$q = -\frac{k\rho g}{2\mu}\frac{\partial h^2}{\partial x}\bigg|_{x=0} = -\frac{k\rho g h_0^2}{2\mu\sqrt{a h_0 t}}\frac{d\varphi^2}{d\xi}\bigg|_{\xi=0}.$$

But according to preceding/previous $\varphi = \zeta'(\sqrt{2}\zeta)$, so that

$$\frac{d\varphi^2}{d\xi}\bigg|_{\xi=0} = 2\left[\zeta'(\eta)\,\zeta''(\eta)\,\frac{d\eta}{d\zeta}\,\frac{1}{\sqrt{2}}\right]_{\eta=0} = \sqrt{2}\,\zeta''(0).$$

as is known frcm boundary-layer theory, value $\zeta''(0)$ through which is expressed the coefficient of friction cf plate, is equal to 0.332, whence we obtain finally expression for a fluid flow, which escape/ensues in reservoir, in the form:

$$q=-0{,}332\,\frac{k\rho g h_0^{3/2}}{\mu\sqrt{2ah_0t}}=-0{,}332\,h_0^{3/2}\sqrt{\frac{k\rho gm}{\mu t}}. \qquad \text{(IV.3.19)}$$

the examined sclution was found by P. Ya. Polubarinova-Kochina [92-94].

MT/ST-76-1860

Pages 89-125.

§4. Axisymmetric self-similar motions in the infinite layer with non-zero initial pressure of gas or the level of liquid.

1. Self-similar isothermal flow of thermodynamically perfect gas with constant viscosity, which appears during pumping or gas bleed through the hole. Let us examine infinite horizontal layer with a power H, revealed according to entire power by the cylindrical hole

whose direction is perpendicular to the direction of the stretch of layer. In the moment layer is saturated by gas, which are located under pressure P. Through revealing the layer hole at the moment begins to be pumped into the gas with constant mass is diverged q. Let us examine the appearing in this case filtration flow of gas.

Since the picture of the motion is symmetrical ad is identical in all planes, perpendicular to the axes of hole, pressure distribution of gas depends only on time t and distance r of the point of layer in question on the axis of hole r = 0 satisfies an equation

$$\frac{\partial p}{\partial t} = a^2 \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial p^2}{\partial r}\right); \quad a^2 = \frac{k}{2m\mu}. \qquad (IV.4.1)$$

The initial pressure of gas in layer constantly is equal to P, so that the initial condition and condition at infinity take the form:

$$p(r, 0), = P, \quad p(\infty, t) = P. \qquad (IV.4.2)$$

Page 90.

Through the hole whose radius is equal to R, into layer is pumped into the gas with the constant mass flow rate of q:

$$-\frac{\pi k H \rho_0}{\mu p_0}\left(r\frac{\partial p^2}{\partial r}\right)_{r=R} = q. \qquad \text{(IV.4.3)}$$

We will count a radius of hole negligible (below we will give the estimations, which justify this assumption). Then condition (IV.4.3) is rewritten in the form:

$$\left(r\frac{\partial p^2}{\partial r}\right)_{r=0} = -\frac{q\mu p_0}{\pi k H \rho_0}, \qquad \text{(IV.4.4)}$$

Thus, the unknown distribution of pressure in layer, which satisfies an equation (IV.4.1) and conditions (IV.4.2) and (IV.4.4), depends on the determining parameters r, t, a^2, $\frac{q\mu p_0}{\pi k H \rho_0}$, of dimensionality of which the following

$[r] = L$; $[t] = T$; $[a^2] = [p]^{-1}L^2T^{-1}$; $\left[\frac{q\mu p_0}{\pi k H \rho_0}\right] = [p]^2$; $[P] = [p]$ ([p] dimensionality of pressure). With the aid of the analysis of dimensionality it is possible to be convinced of the self-similarity of motion in question. Pressure distribution in this case is represented in the form:

$$p = PF_1(\xi, \lambda); \quad \xi = \frac{r}{\sqrt{a^2Pt}}; \quad \lambda = \frac{q\mu p_0}{\pi k H \rho_0}. \qquad \text{(IV.4.5)}$$

By substituting (IV.4.5) into equation (IV, 4.1) and conditions (IV.4.2) and (IV 4.4), we will obtain that the function $F_1(\xi, \lambda)$

is the integral of equation

$$\frac{d^2F_1^2}{d\xi^2}+\frac{1}{\xi}\frac{dF_1^2}{d\xi}+\frac{\xi}{2}\frac{dF_1}{d\xi}=0 \qquad \text{(IV.4.6)}$$

under boundary conditions

$$\left(\xi\frac{dF_1^2}{d\xi}\right)_{\xi=0}=-\lambda;\quad F_1(\infty,\lambda)=1. \qquad \text{(IV.4.7)}$$

The qualitative picture of the location of the integral curves of equation (IV.4.6) is investigated with analogous that as described in §1: in exactly the same manner the order of equation (IV.4.6) is reduced to the first, then is investigated the picture of the integral curves of first-order equation, whereupon results are transferred by integral curve equations (IV.4.6). This investigation shows that integral curve equations (IV.4.6), which satisfy the second condition (IV.4.7), fall into two classes divided between themselves integral curve $F_1(\xi,0)\equiv 1$, ppropriate, as can easily be seen, $\lambda = 0$ (Fig. IV.12). The curves of the first class,

which are arrange/located above the curve $F_1(\xi, 0) \equiv 1$, boundless closely approach the axis ordinate, asymptotically departing into infinity during a decrease in the ξ down to zero. With the $\xi \to 0$ of function F_1 the $F_1(\xi, \lambda)$ slowly grow/rises according to the law

$$F_1(\xi, \lambda) = \sqrt{-\lambda \ln \xi} + 0(1), \qquad \text{(IV.4.8)}$$

so that to each of the integral curves of the first class it corresponds its value of the parameter λ, which monotonically grow/rises from zero to with distance from the curve $F_1(\xi, 0) \equiv 1$.

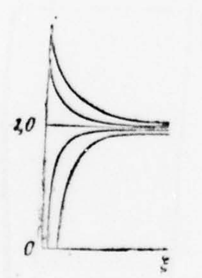


Fig. IV.12.

AD-A044 774 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 13/11

THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS (TEORIY--ETC(U)

JAN 77 G I BARENBLATT, V M YENTOV, V M RYZHIK

UNCLASSIFIED FTD-ID(RS)T-1860-76-PT-1 NL

4 OF 5

AD A044774

With the $\xi \rightarrow \infty$ the ordinates of the curves of both classes rapidly they approach unity according to the law

$$F_1(\xi, \lambda) = 1 + 0\left[\frac{1}{\xi}\exp\left(-\frac{\xi^2}{8}\right)\right]. \qquad \text{(IV.4.9)}$$

. The curves of the second class, which are arrange/located under integral curve $F_1(\xi, 0) = 1,$ do not reach the axis of ordinates, but they conclude, approaching square with the axis of abscissas [which is itself by the line of equation (IV.4.6), since on this line of transforms into zero the coefficient of higher derivative in this equation]. In this case, instead of the first condition which satisfy all the integral curves of the first class, which correspond λ > to 0, these curves satisfy condition

$$\left(\xi \frac{dF_1^*}{d\xi}\right)_{\xi=\bar{\xi}(\lambda)} = -\lambda, \qquad \text{(IV.4.10)}$$

where the $\bar{\xi}(\lambda)$ - the coordinate of the point of intersection of the curve in question with the axis of abscissas. To each curve

corresponds the determined value λ, which monotonically decreases with the removal/distance of curves from the integral curve of $F_1(\xi, 0) \equiv 1$ from zero to $-\infty$.

The integral curves of the second class describe the self-similar motions, in process of which occurs not the gas injection in plates, as in the case of the motions, which correspond the integral curves of the first class (λ > 0), but gas bleed from layer with the flow rate, determined by the corresponding this curved value λ:

$$q = \frac{\lambda \pi k H P^2 \rho_0}{\mu p_0}. \qquad \text{(IV.4.11)}$$

(in this formula the mass flow rate q is considered negative).

It should be noted that, by creating a sufficient pressure differential, it is possible, in principle, to pump into gas into layer with any large flow rate through the hole as small as desired radius. However, to take/select gas from layer possible only with the

flow rates, which do not exceed that flow rate which corresponds to establishment of the wall of the hole of zero pressure. A further increase in the gas flow of the take/selected gas is possible only under the condition of the expansion of hole. Thus, unlike the case of pumping of gas, it is not possible to pose the problem of gas bleed through the hole of a negligible radius. Page 91.

Curved $F_1(\xi, \lambda)$ with $\lambda > 0$ (curves the second, class) corresponds to the self-simulation motions in which the gas bleed with constant flow rate/consumption, determined by formula (IV.4.11), occurs through the being expanded hole whose radius increases according to the law

$$R = \bar{\xi}(\lambda) \sqrt{a^2 Pt}, \qquad \text{(IV.4.12)}$$

whereupon of the wall of this being expanded hole pressure constantly and is equal to zero. Let us note that this expansion of the selective hole in no way impedes the application/use of the solutions in question to practical tasks, since for the values of the parameter λ, which represent practical interest, this fictitious hole, as show carried out chalchulation), always it will be located within the present hole.

Further, if we take part of any integral curve $F_1(\xi, \lambda)$, by that belonging to the first either second class, from certain $\xi = \eta \gg \bar{\xi}(\lambda)$ to $\xi = \infty$, that formula (IV.4.5) will be the self-similar solution of the problem, which corresponds to constant initial pressure and selection or gas injection (depending on sign λ) through the being expanded hole. In this case, in the wall of well, is supported the constant pressure, equal to

$$PF_1(\eta, \lambda). \qquad \text{(IV.4.13)}$$

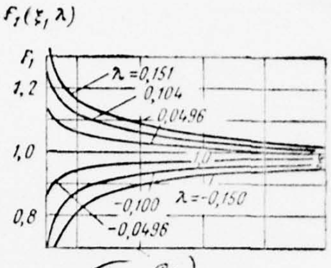
Radius R of the being expanded hole increases according to the law

$$R = \eta \sqrt{a^2 Pt}. \qquad \text{(IV.4.14)}$$

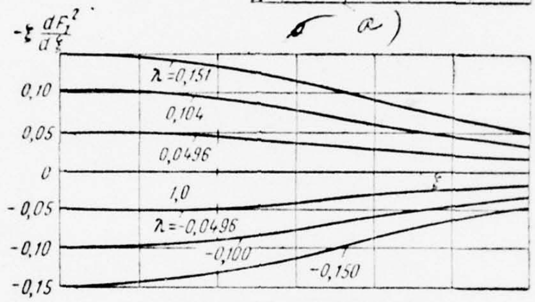
For a following presentation it is useful to explain, which order of magnitude of $\lambda = q\mu p_0 [\pi k \rho_0 H P^2]^{-1}$ is encountered in practical tasks. Let us take as an example the case for which value λ

will be very high, by this very it will be determined the order of the upper limit of value λ. Let through the hole be take/selected 1,000,000 m^3 of gas per day (is in form a volume with atmospheric conditions); this flow rate is sufficiently high. Let, further, the viscosity of the gas of μ be equal to 0.01 cp, the permeability of the porous medium k = 1 d = 10^{-8} cm^2, thickness of layer H = 10 m, the initial sheet pressure P = 30 kgf/cm^2 (relatively small pressure for so high a gas bleed); for values p_0 and ρ_0 let us take respectively 1 kgf/cm^2 and the density of gas at pressure 1 kgf/cm^2, so that value q/ρ_0, is represented by the assigned volumetric flow rate of the gas, take/selected through the hole. Transfer/converting to the identical units of measurement and substituting the led values of the parameters in expression for λ, let us accept $\lambda \approx 0.04$. It turned out tat in the real cases the parameter λ was equal to 0.01-0.02 and less.

Figure IV. 13a, b depicts curved $F_1(\xi, \lambda)$, corresponding to several values of the parameter λ both positive and negative, and also the corresponding curves are an $\xi dF_1^*/d\xi$.


$F_1(\xi, \lambda)$
F_1
1,2
1,0
0,8
$\lambda = 0,151$
0,104
0,0496
1,0
ξ
-0,100
$\lambda = -0,150$
-0,0496

a)


$-\xi \frac{dF_1^2}{d\xi}$
0,10
0,05
0
-0,05
-0,10
-0,15
$\lambda = 0,151$
0,104
0,0496
1,0
ξ
$\lambda = -0,0496$
-0,100
-0,150

b)

Fig. IV.13.

These curves show that in sufficiently considerable domain near the point of $\xi=0$ (respectively near $\xi=\bar{\xi}(\lambda)$ for curves, that correspond $\lambda<0$) function - $\xi dF_1^2/d\xi$ is close to its value with of $\xi=0$ (respectively with $\xi=\bar{\xi}(\lambda)$), i.e. to λ. In this case, the basic change in function $F_1(\xi,\lambda)$, i.e. the basic change in the pressure of gas, is concentrated precisely in this domain. At the same values of ξ, for which, the function, $\xi dF_1^2/d\xi$ already substantially is deflect/diverted but to value from λ, function $F_1(\xi,\lambda)$ proves to be sufficient close to unity. In the virtually most interesting range of values of the parameter λ, equal in absolute value of one hundredth and less, this property of the constancy of function - $\xi dF_1^2/d\xi$ in the range where $F_1(\xi,\lambda)$ differs significantly from unity, it is expressed even more sharply. Table IV.4 gives the results of the numerical calculation of the curves $F_1(\xi,\lambda)$, appropriate λ = -0.009999 and λ = -0.004994. Through the ξ_* is deisgnated the value of the argument of ξ, possessing that property, that with the $\xi<\xi_*$ of value - $\xi dF_1^2/d\xi$ they differ from λ less than to 0.01o/o. That means with $\xi\leqslant\xi^*$ with this same degree of accuracy is fulfilled the relationship/ratio

$$\xi\frac{dF_1^2}{d\xi}=-\lambda, \qquad \text{(IV.4.15)}$$

whence with $\xi<\xi_*$ we have with the same degree of accuracy, with which are calculated the tables,

$$F_1^2(\xi,\lambda)-F_1^2(\xi_*,\lambda)=-\lambda\ln\frac{\xi}{\xi_*}. \qquad \text{(IV.4.16)}$$

Therefore values $F_1(\xi, \lambda)$ with of $\xi < \xi_*$ Table IV.4 gives. The numerical calculations conducted they show as this evident from Table IV.4 that when $|\lambda| < 0{,}01$ value $F_1(\xi_*, \lambda)$ differs from unity less than by 0.03, so that with $\xi \geqslant \xi_*$ is correct inequality $1 > F_1(\xi, \lambda) > 0{,}97.$ Hence it follows that with virtually completely sufficient accuracy in this range equation (IV.4.6) for function $F_1(\xi, \lambda)$ can be replaced with linear relative to $F_1^2(\xi, \lambda)$ equation

$$\frac{1}{\xi} \frac{d}{d\xi} \xi \frac{dF_1^2}{d\xi} + \frac{\xi}{4} \frac{dF_1^2}{d\xi} = 0 \qquad \text{(IV.4.17)}$$

[in the last/latter term of equation (IV.4.6) is added factor $F_1(\xi, \lambda)$, according to the preceding differing little from unity]. This linear equation easily is integrated and gives

$$\xi \frac{dF_1^2}{d\xi} = C e^{-\xi^2/8}, \qquad \text{(IV.4.18)}$$

where C - the constant of integration. Let us determine this constant from the condition that with $\xi = \xi^*$ are value the $\xi \frac{dF_1^2}{d\xi} = -\lambda.$ We have

$$C = -\lambda \exp(-{}^1/_8 \xi_*^2).$$

Table IV-4.

$\lambda = -0{,}009999$			$\lambda = -0{,}004994$		
ξ	$F_1(\xi, \lambda)$	$\xi \frac{dF_1^2}{d\xi}$	ξ	$F_1(\xi, \lambda)$	$\xi \frac{dF_1^2}{d\xi}$
$\xi_* =$ 0,005787	0,9701	0,009999	$\xi_* =$ 0,003886	0,9842	0,004994
0,01157	0,9737	0,009999	0,01555	0,9877	0,004994
0,01929	0,9763	0,009999	0,03109	0,9894	0,004993
0,03472	0,9793	0,009998	0,06218	0,9912	0,004992
0,06558	0,9825	0,009994	0,1244	0,9929	0,004984
0,09645	0,9845	0,009987	0,2487	0,9947	0,004955
0,1582	0,9870	0,009968	0,4974	0,9964	0,004841
0,2816	0,9899	0,009899	0,9949	0,9980	0,004412
0,5285	0,9930	0,009653	1,492	0,9988	0,003779
0,7754	0,9948	0,009270	2,487	0,9996	0,002305
1,269	0,9970	0,008167	3,482	0,9999	0,001098
1,763	0,9982	0,006770	—	—	—
2,751	0,9994	0,003879			
3,738	0,9999	0,001743			

. Since for the virtually interesting range of $|\lambda| < 0.01$ in question value of ξ^* is very small (<0.01) and an $e^{-1/4\xi_*^2}$ it differs from unity not more than in the sixth decimal point, it is possible to set/assume $C = -\lambda$.

Page 95.

Integrating the equation (IV.4.18) again, we obtain with $\xi \geq \xi_*$

$$F_1^2(\xi, \lambda) = D - \lambda \int \frac{1}{\xi} e^{-\xi^2/4} d\xi. \qquad \text{(IV.4.19)}$$

The constant of integration D is located from the condition that with $\xi = \xi_*$ value $F_1(\xi_*, \lambda)$, corresponding to the solution to linear equation (IV.4.17), coincides with value $F_1(\xi, \lambda)$, obtained by numerical integration. This gives with $\xi \geqslant \xi_*$.

$$F_1^2(\xi, \lambda) = F_1^2(\xi_*, \lambda) + \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi_*^2}{8}\right) - \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi^2}{8}\right), \quad \text{(IV.4.20)}$$

where Ei is exponential integral.

With $\xi \to \infty$ the function, determined by equation (IV.4.20), strictly speaking, does not satisfy the second condition (IV.4.7); with $\xi \to \infty$ this function it approaches not unity, but for value

$$S = \sqrt{F_1^2(\xi_*, \lambda) + \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi_*^2}{8}\right)}.$$

However, it proves to be that this value very differs little from unity, difference for the virtually most interesting range of $|\lambda| < 0.01$ is located of 0.01o/o. So with $\lambda = -0.009999$ we have:

$\xi_* = 0.005787$; $-\frac{1}{2}\mathrm{Ei}\left(\frac{1}{8}\xi_*^2\right) = 5{,}905$; $F_1(\xi_*, \lambda) = 0{,}9701$;

S = 1.00007. For less in absolute value λ this difference is still less, so that value S for this range λ can be accepted equal to unity. Thus, the constructed by us solution to linear equation satisfies with completely sufficient degree of accuracy the second condition (IV.4.7).

Thus, with $\xi \geq \xi_*$ the function of $F_1(\xi, \lambda)$ with an accuracy to 0.01o/o is represented in the form:

$$F_1(\xi, \lambda) = \sqrt{1 - \frac{\lambda}{2}\mathrm{Ei}\left(-\frac{\xi^2}{8}\right)}. \qquad \text{(IV.4.21)}$$

Let us note now that since in the interval of $|\lambda| < 0{,}01$ in question the value of ξ_* is very low, for the function of the $\mathrm{Ei}\left(-\frac{\xi^2}{8}\right)$ with cf $\xi \leqslant \xi_*$ with high precision, is fulfilled asymptotic formula

$$\mathrm{Ei}\left(-\frac{\xi^2}{8}\right) = -\ln\frac{8}{\gamma\xi^2},$$

(for example, see Yanke and Emde [129]). Therefore with $\xi \leqslant \xi_*$ with the accuracy, greater than 0.01o/o, occurs equality

$$\lambda \ln\frac{\xi}{\xi_*} = \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi^2}{8}\right) - \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi_*^2}{8}\right).$$

Page 96.

Substituting this relationship/ratio in formula (IV.4.16), we obtain, that with $\xi \leqslant \xi_*$ with an accuracy to 0.01o/o

$$F_1^0(\xi, \lambda) = F_1^0(\xi_*\lambda) + \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi_*^2}{8}\right) - \frac{\lambda}{2}\,\mathrm{Ei}\left(-\frac{\xi^2}{8}\right).$$

But this expression coincides precisely with the relationship/ratio (IV.4.20), which the function of $F_1(\xi, \lambda)$ satisfies with $\xi \leqslant \xi_*$. Furthermcre, above it was shown that the first two terms of the right side of the preceding/previous formula in sum with high precision were equal to unity. Hence follows very essential the conclusion that in the virtually most interesting interval of the values of the parameter λ, λ < 0.01 the function of $F_1(\xi, \lambda)$ is represented in the form (IV.4.21) at all values of ξ.

Transfer/converting from the function of $F_1(\xi, \lambda)$ to pressure P according to formula (I.4.5), we obtain, that for

$$|\lambda| = \left|\frac{q\mu p_0}{\pi k H \rho_0 P^2}\right| < 0{,}01$$

pressure distribution with very high degree of accuracy is represented for all values of r and t the form:

$$p^2 = P^2 - \frac{q\mu p_0}{2\pi k H \rho_0} \operatorname{Ei}\left(-\frac{r^2}{8a^2 P t}\right). \qquad \text{(IV.4.22)}$$

Specifically, such would be obtained the solution of problem, if we were replaced in the equation (IV.4.1), which it is possible to present in the form

$$r\frac{\partial p^2}{\partial t}-2a^2p\frac{\partial}{\partial r}r\frac{\partial p^2}{\partial r}=0, \qquad \text{(IV.4.1*)}$$

factor p in the second term at value of p = P of this factor with r = ∞, i.e., if from equation (IV.4.1) under the same boundary and initial conditions send to linear relative to p^2 equation

$$r\frac{\partial p^2}{\partial t}=2a^2P\frac{\partial}{\partial r}r\frac{\partial p^2}{\partial r}. \qquad \text{(IV.4.23)}$$

This method of the linearization of equation (IV.4.1) was proposed for the first time by L. S. Leybenzon [71]. The given calculations show virtually the register of the solution of the nonlinear axisymmetric problem in question with the solution of the linearized problem. The success of linearization is explained in this case by the fact that in the case of axisymmetric motions the range of motion is divide/marked off into two parts: 1) the range of quasi-stationary motion, which corresponds to low values of ξ, in which is concentrated basic part of entire the pressure differential, but the flow of gas is almost constant, and 2) - the range of small depressions (the pressure differentials), in which the flow of gas comparatively slowly decreases, and the pressure differentials are small.

Page 97.

In the range of quasi-stationary motion, not only a difference in the values of $r\frac{\partial p^2}{\partial t}$ and $2a^2p\frac{\partial}{\partial r}r\frac{\partial p^2}{\partial r}$ is equal to zero, as this follows from equation (IV.4.1), but also each of these values itself is vanishingly small (comparatively with the values of these quantities at those points, where they are maximum). Therefore in this range the flow of gas, equal - $\frac{\pi kH\rho_0}{\mu p_0}\left(r\frac{\partial p^2}{\partial r}\right)$, is almost constant, and factor value at the second member (IV.4.1) is insignificant, and with a high degree of accuracy it is possible to replace in this factor p (r,t) by P (r,t). In the range of small depressions, in the determined part which both members (IV.4.1) differ significantly from zero, the possibility of this replacement is caused by the smallness of difference p (r,t) - P.

The discovered admissibility of linearization with the description of the nonlinear axisymmetrical motions not of dependence on the value of the appearing pressure differential makes it possible to make important conclusions in connection with the more common/general/total classes of motion.

Let us note now that in real tasks the flow of the gas through the hole although small; but the final fixed radius so that boundary condition on hole on the basis of expression (IV.4.3) takes the form:

$$\left(r\frac{\partial p^2}{\partial r}\right)_{r=R} = -\frac{q\mu p_0}{\pi k H \rho_0}. \qquad \text{(IV.4.24)}$$

Let us show that the constructed above self-similar solution satisfies with a high degree of accuracy this condition already after several seconds after the beginning of process.

In fact, on the basis (IV.4.5) we have

$$\left(r\frac{\partial p^2}{\partial r}\right)_{r=R} = P^2\left(\xi\frac{dF_1^2}{d\xi}\right)_{\xi=R/\sqrt{a^2Pt}}. \qquad \text{(IV.4.25)}$$

The numerical calculations, carried out for a curve λ = -0.009999, shows (see Table IV.4), that already with $\xi = 0{,}1582$ the value of the function of $\xi dF_1^*/d\xi$ is equal to 0.009968, i.e., differs from λ are less than to 0.5o/o and still less with less than the ξ.

At a radius of hole $R \approx 10$ cm, permeability $k \approx g = 10^{-8}$ cm^2, porosity $m \approx 0.2$, the viscosity of $\mu \approx 10^4$ g/cm·s, the value of $a^2P = \frac{kP}{2m\mu}$ has an order of 10^3-10^4 cm^2/s, and then already with t = 3s

$$\xi = R/\sqrt{a^2Pt} < 0{,}19.$$

Therefore it is possible with very high degree of accuracy to set/assume with t > 3 s

$$\left(\xi \frac{dF_1^*}{d\xi}\right)_{\xi=R/\sqrt{a^*Pt}} = -\lambda.$$

Page 98.

Utilizing this fact in relationship/ratio (IV.4.25), we obtain, that after several seconds after motion self-similar solution (Iv.4.5) with a high degree of accuracy satisfy an equation

$$\left(-r\frac{\partial p^2}{\partial r}\right)_{r=R}=\lambda P^2=\frac{q\mu p_0}{\pi k H \rho_0},$$

i.e. to boundary condition (IV.4.24).

As it was shown above, being encountered in practice values of the parameter λ on module/modulus is considerably less than examined recently the value, approximately equal is 0.08. Therefore for less λ condition (IV.4.25) will be satisfied still faster.

Above it was noted that the self-similar solutions in $\lambda < 0$ correspond to gas bleed from the layer through the being expanded with time hole. Let us show now that this abnormal, at first glance,

the property of solutions does not prevent their application/use to real tasks, since for that which is of practical interest of time the being expanded (fictitious) hole always remains within the present hole. For this, let us determine the order of magnitude of $\bar{\xi}(\lambda)$ - the coordinate of the points of the approach of the curve of $F_1(\xi, \lambda)$ at $\lambda < 0$ to the axis of abscissas. As was noted above, with $\xi \leqslant \xi_*$, i.e. specifically, with $\xi = \bar{\xi}(\lambda)$, function $F_1(\xi, \lambda)$ with high degree of accuracy satisfies relationship (IV.4.16):

$$F_1^2(\xi, \lambda) - F_1^2(\xi_*, \lambda) = -\lambda \ln \frac{\xi}{\xi_*}.$$

Set/assuming in this relationship/ratio of the $\xi = \bar{\xi}(\lambda)$, $F_1(\xi, \lambda) = 0$, we obtain

$$F_1^2(\xi_*, \lambda) = \lambda \ln \frac{\bar{\xi}(\lambda)}{\xi_*}; \quad \bar{\xi}(\lambda) = \xi_* e^{\frac{1}{\lambda} F_1^2(\xi_*, \lambda)}.$$

With $\lambda = -0{,}08$, $F_1(\xi_*, \lambda) \approx 0{,}72$ are value the $\xi_* = 0{,}0050$

whence of $\bar{\xi}(\lambda) \approx 0{,}005\,e^{-6{,}5} = 0{,}75 \cdot 10^{-5}$. . As shows formula (IV.4.12), time interval T for which the being expanded internal hole reaches the size/dimensions of the present hole, it comprises

$$T = \frac{R^2}{\bar{\xi}^2 a^2 P},$$

that on the strength of the preceding/previous estimations for an $\bar{\xi}$, a^2P and R gives approximately T = 2•10⁸ s - about six years. Let us note that the value λ = - 0.08 is very great comparatively with values, which are encountered in practice. During decrease λ, value T sharply grow/rises: so, with λ = -0.01 $T \approx 10^{75}$ years. Thus, for real tasks the being expanded (fictitious) hole always remains within present.

the given estimations show that the self-similar solution in question is completely suitable for real tasks.

The self-similarity under the present heading of the task in question was noted by L. S. Leybenzon [72] and by p. Ya. Polubarionova-Kochina [94].

Page 99.

The solution of this problem outlined above is given by G. I. Barenblatt [12, 9]. Numerical calculations were carried out under N. P. Trifonova's management/manual [24].

§5. Some special self-similar motions.

In this paragraph will be presented the solutions to some self-similar problems of unsteady filtration, which are of special interest. In connection with the fact that the systematic side of the construction of similar solutions is sufficiently explained in the preceding/previous paragraphs, presentation here shorter: the reader, interested in the details of computation it will be able to find them in cited literature.

1. Lift of liquid level with the cessation of filtration into empty reservoir and with the attack of the leading edge of liquid for

cbstruction. Let to moment t = o correspond the stationary distribution of liquid level, which corresponds the flat free-flow escape of liquid from layer intc empty reservoir. If the second boundary of layer is located sufficiently far, then layer can be considered semi-infinite; the initial distribution of liquid level of $h_0(x)$, satisfying an equation (IV.3.1) and to condition h_0 (0) = 0, is represented in the form:

$$h(x,\ 0)=h_0(x)=\sqrt{\frac{2q\mu x}{k\rho g}}, \qquad \text{(IV.5.1)}$$

where q is the direct flow of the liquid, which escape/ensues of layer. Let us note that the increase of floating surface with increase in x infinite grow/rises; however this is not important, since, examining infinite layer, we are interested the only to initial stage of the motion when the disturbance/perturbations of steady state, produced near boundary of x = 0, insignificantly is expressed near the second boundary.

Let us assume now that at zero time the boundary of layer x = 0 suddenly is insulated, so that the flow of the liquid through it

ceases. Let us examine subsequent process of the lift of liquid level. Fluid flow on boundary with x = 0 is equal to zero; this gives condition

$$\frac{\partial h^2}{\partial x}=0 \qquad (t>0). \tag{IV.5.2}$$

As it follows from dimensional analysis, the solution to equation (IV.3.1) under the conditions (IV.5.1) and (IV.5.2) self-similarly is represented in the form:

$$h=\sqrt{\frac{2q\mu x}{k\rho g}}\,G(\zeta);\quad \zeta=\frac{x}{\left(at\sqrt{\frac{2q\mu}{k\rho g}}\right)^{2/3}}=\frac{x}{\left(\frac{t}{m}\sqrt{\frac{k\rho g q}{2\mu}}\right)^{2/3}}. \tag{IV.5.3}$$

Let us place $G(\zeta)=\zeta^{-1/2}g(\xi);\ \xi=\zeta\sqrt{\frac{4}{3}}$. Then in accordance with equation (IV.3.1) and conditions (IV.5.1) and (IV.5.2) function g of $g(\xi)$ satisfies an equation

$$\frac{d^2g^2}{d\xi^2}+\frac{1}{2}\xi\frac{dg}{d\xi}-\frac{1}{4}g=0, \qquad \text{(IV.5.4)}$$

ccinciding with equation (IV.1.7) with $\lambda = 1/4$, under boundary conditions

$$\frac{dg^2}{d\xi}\Big|_{\xi=0}=0,\quad \lim_{\xi\to\infty}\frac{g(\xi)}{\sqrt{\xi}}=\sqrt[4]{\frac{3}{4}}. \qquad \text{(IV.5.5)}$$

Let us construct the function of $g_0(\xi)$ - the solution of the problem of Cauchy for equation (IV.5.4), that satisfies conditions $g_0(0)=1,\ g_0'(0)=0$.

As it shows the investigation, led into §1, with $\xi\to\infty$ for the function of $g_0(\xi)$, that which belong to integral curves I class of equation (IV.5.4), correctly relationship/ratio $\lim\limits_{\xi\to\infty} g_0(\xi)/\sqrt{\xi}=\text{const}$; according to calculation the constant is equal 0,7772, and the function of $g_0(\xi)$ satisfies an equation (IV.5.4) and the first condition (IV.3.5), but does not satisfy the second condition (IV.3.5). Since the function of $\mu^{-2}g_0(\mu\xi)$ satisfies an equation (IV.5.4) and the first condition (IV.5.5) with any μ, that selecting $\mu=\mu_*=0,8763$, we obtain, that the function of $g(\xi)=\mu_*^{-2}g_0(\mu_*\xi)$ satisfies all conditions of task, and the solution is represented in the form:

$$h=\sqrt[3]{\frac{2q^2\mu t}{mk\rho g}}\,g\left[\frac{2x\sqrt{3}}{\left(\frac{t}{m}\sqrt{\frac{k\rho gq}{2\mu}}\right)^{2/3}}\right]=1,260\sqrt[3]{\frac{q^2\mu t}{mk\rho g}}\,g\left[\frac{1,633x}{\left(\frac{t}{m}\sqrt{\frac{k\rho gq}{\mu}}\right)^{2/3}}\right]. \qquad \text{(IV.5.6)}$$

Values of the function of $g(\xi)$ are given in Fig. IV.13. Specifically, with x = 0, i.e., by the boundary of layer, level of

liquid it grow/rises in the course of time according to the law

$$h(0,\ t) = 1{,}641 \sqrt[3]{\frac{q^2\mu t}{mk\rho g}}. \qquad \text{(IV.5.7)}$$

Examples of self-similar motions examined above show that the leading edge of liquid is spread on dry confining stratum at the final velocity, the floating surface approaching the point of its contact with confining stratum at acute angle and near this point has a form of inclined plane.

The following reasoning, which belongs Ya. to B. Zeldovich and A. S. Kompaneyts [50], shows that this fact occurs, also, in the general case of the motion of liquid in dry soil. In fact, the equation (IV.3.1), which describes the distribution of liquid level, can be rewritten as follows:

$$\frac{\partial h}{\partial t} = a^2\left[2\left(\frac{\partial h}{\partial x}\right)^2 + 2h\frac{\partial^2 h}{\partial x^2}\right]. \qquad \text{(IV.5.8)}$$

. Page 101.

But at point x_0 (t) - with the leading edge of the range, occupied with liquid, is fulfilled the relationship/ratic of $h[x_0(t), t] = 0$. For a certainty we will consider that the front moves from right to left. Then, differentiating the last/latter relationship/ratio, we find

$$\frac{\partial h}{\partial t}+\left(\frac{\partial h}{\partial x}\right)_{x=x_0}\frac{dx_0}{dt}=0;\quad \frac{\partial h}{\partial t}=\left(\frac{\partial h}{\partial x}\right)_{x=x_0}v_0,$$

where the v_0 - the absolute instantaneous speed of the movement of the boundary of this range (speed itself is negative). Substituting these relationship/ratios into equation (IV.5.8), we find

$$v_0\left(\frac{\partial h}{\partial x}\right)_{x=x_0}=2a^2\left(\frac{\partial h}{\partial x}\right)^2_{x=x_0};\quad \left(\frac{\partial h}{\partial x}\right)_{x=x_0}=\frac{v_0}{2a^2}=\frac{v_0 m\mu}{k\rho g}. \quad \text{(IV.5.9)}$$

Hence it is obtained, that near leading edge the distribution of liquid level takes the form:

$$h=\frac{v_0 m\mu}{k\rho g}(x_0-x), \quad \text{(IV.5.10)}$$

whereupon during brief time intervals it is possible the speed of the displacement/movement of leading edge tc consider constant.

Thus, we come to the following formulation of the problem.

At the moment the distribution of liquid level during x > 0 is expressed as

$$h(x,\ 0)=\frac{v_0 m\mu x}{k\rho g}. \qquad (IV.5.11)$$

On boundary of x = 0, which corresponds to impenetrable obstruction, fluid flow is equal to zero:

$$\left.\frac{\partial h^2}{\partial x}\right|_{x=0}=0. \qquad (IV.5.12)$$

The distribution of the level of liquid h (x, t), that satisfies an equation (IV.3.1) and conditions (IV.5.11) and (IV.5.12), self-similarly can be is represented in the form:

$$h=\frac{v_0^2 m\mu t}{2k\rho g}u(\xi);\quad \xi=\frac{2\sqrt{2}\,x}{v_0 t}. \qquad \text{(IV.5.13)}$$

Function $u(\xi)$ satisfies an equation

$$\frac{d^2u^2}{d\xi^2}+\frac{1}{2}\xi\frac{du}{d\xi}-\frac{1}{2}u=0 \qquad \text{(IV.5.14)}$$

under boundary conditions

$$\left(\frac{du^2}{d\xi}\right)_{\xi=0}=0;\quad \lim_{\xi\to\infty}\frac{u(\xi)}{\xi}=\frac{1}{\sqrt{2}}. \qquad \text{(IV.5.15)}$$

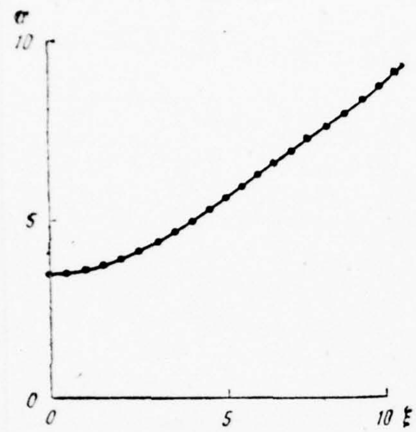


Fig. IV.14.

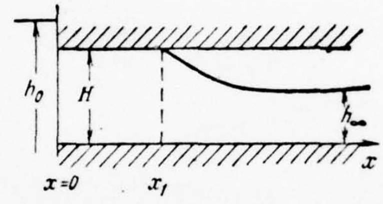


Fig. IV.15.

The solution of the $u(\xi)$ of boundary-value problem (IV.5.14) - (IV.5.15) is expressed as in the preceding case, through the solution of the problem of Cauchy for equation (IV.5.14), the satisfying conditions $u_0(0)=1, u_0'(0)=0.$ Values of the function of $u(\xi)$ are given in Fig. IV.14. Specifically, liquid level during the obstruction itself, i.e., with x = 0, it grow/rises according to the law

$$h(0,\ t)=1,79\frac{m\mu t}{k\rho g}. \qquad \text{(IV.5.16)}$$

2. Pressure-free-flow motion on the zero initial level of liquid. Let us examine the uniform layer of the final power H. Assume that at first layer was filled by motionless liquid to the level of $h_\infty < H$; at torque/moment t = 0 on the boundary of layer x = 0 it is created pressure head $h_0 > H$. The distribution of confining stratum at the subsequent points in time will, obviously, take the form, shown in Fig. IV.15. In part of the layer, which directly adjoins the initial section x = 0, the pressure head will exceed value H. Therefore layer will be filled by liquid wholly, and motion on this section will be pressure. At certain point $x = x_1$, the pressure head will be equalized with H and with $x > x_1$ motion it will

become without barrier. If the layer did not contain at first the liquid of $(h_\infty = 0)$, that the zone of motion it is spread at finite speed; if the initial level differs from zero, then motion immediately seizes entire layer. The formulated problem has the self-similar solution:

$$h = Hf_0(\xi); \quad \xi = x\sqrt{\frac{m\mu}{kH\rho gt}}. \qquad \text{(IV.5.17)}$$

. In this case, from $\xi = 0$ to $\xi_1 = x_1\left(\frac{m\mu}{kH\rho gt}\right)^{1/2}$ motion is a barrier, and from $\xi = \xi_1$ and further - free-flow. By At zero initial level of liquid the range of free-flow movement stretches to certain finite value

$$\xi = \xi_2 = x_2\left(\frac{m\mu}{kH\rho gt}\right)^{1/2}.$$

MT/ST-76-1860

Page 103.

The reader easily will finish the construction of the self-similar solution of the problem of pressure-free-flow motion in the zero initial level of liquid, utilizing for the effective construction of the solution the self-similar solutions § 1.

3. Solutions of the type of instantaneous sources for the problems of the polytropic filtration of thermodynamically ideal gas.

Let in the infinite volume of the porous medium occur the filtration of gas in polytropic communication/connection of density and pressure of the filtering gas. Assuming to be motion one-dimensional, we have equation for the density of gas in the form:

$$\frac{\partial \rho}{\partial t} = a^2 \frac{1}{r^s} \frac{\partial}{\partial r}\left(r^s \frac{\partial \rho^{n+1}}{\partial r}\right); \quad a^2 = \frac{k n p_*}{\mu m (n+1) \rho_*^n}; \quad \text{(IV.5.18)}$$

$$p = \frac{p_*}{\rho_*^n} \rho^n,$$

where r is space coordinate, i.e., the distance of the point of the porous medium in question from reference plane with movement of plane waves, the distance of this point from the axis of the symmetry of motion - during the axisymmetric flow of gas and its distance from symmetry center - with central-symmetrical motion of gas, but s respectively equal to zero, unity or two for these three types of the symmetry of motion. The initial pressure and the density of gas we assume to be negligibly small in an entire region of the porous medium, so that an initial condition and a condition at infinity they have for the group of problems in question the form:

$$\rho(r, 0) \equiv 0; \quad \rho(\infty, t) = 0. \quad \text{(IV.5.19)}$$

. The solutions stated below correspond to "instantaneous" sources. For the flow of gas by plane waves this means that n the moment certain mass of gas is concentrated near the datum plane r = 0. For axisymmetric and centrally symmetric motions this means that certain mass of gas is concentrated at the moment near axis or, correspondingly, the symmetry center, to which also corresponds value r = 0. Since in the time of motion do not occur any processes, which lead to disappearance or appearance of gas, must be fulfilled some relationship/ratios, which express the preservation/retention/maintaining of the total mass of gas in all volume of the porous medium; these relationships are record/written in the form:

$$\int_0^\infty \rho m\, dr = M_0; \quad \int_0^\infty \rho m r\, dr = \frac{M_1}{2\pi}; \quad \int_0^\infty \rho m r^2\, dr = \frac{M_2}{4\pi}. \quad \text{(IV.5.20)}$$

The solutions in question they represent, for example, for the case of the flow of gas by plane waves, the convenient schematization of the real motions, which appear in simple medium, when the determined mass of gas is concentrated under the large pressure, which considerably exceeds pressure by the remaining points of the porous medium, and then spreads on layer.

Page 104.

Multiplying both parts of the equation (IV.5.18) to r^s and integrating from r = 0 to r = ∞, using conditions ~~(IV.5.18)~~ (II.1,3) at infinity, proportional $\left(r^s \frac{\partial p^{n+1}}{\partial r}\right)_{r\to\infty}$, is equal to zero [1], we obtain

$$\left(r^s \frac{\partial p^{n+1}}{\partial r}\right)_{r=0} = 0. \qquad \text{(IV.5.21)}$$

. FOOTNOTE [1]. Otherwise would not be satisfied the condition with infinity - the second condition (IV.5.19). ENDFOOTNOTE.

Using dimensional consideration, we will obtain the following expressions for gas density during the axisymmetric and centrally symmetric motion of it by the plane waves:

$$\rho = \left(\frac{M_0^2}{a^2 m^2 t} \right)^{\frac{1}{n+2}} f_0 \left[\frac{r}{\left(\frac{M_0^n a^2 t}{m^n} \right)^{\frac{1}{n+2}}} \right]; \qquad \text{(IV.5.22)}$$

$$\rho = \left(\frac{M_1}{2\pi a^2 m t} \right)^{\frac{1}{n+1}} f_1 \left[\frac{r}{\left(\frac{M_1^n a^2 t}{(2\pi m)^n} \right)^{\frac{1}{2n+2}}} \right]; \qquad \text{(IV.5.23)}$$

$$\rho = \left(\frac{M_2^{2/3}}{(4\pi m)^{2/3} a^2 t} \right)^{\frac{3}{3n+2}} f_2 \left[\frac{r}{\left(\frac{M_2^n a^2 t}{(4\pi m)^n} \right)^{\frac{1}{3n+3}}} \right]. \qquad \text{(IV.5.24)}$$

Here the function of $f_s(\xi)$ [ξ it designates in each case its dimensionless argument of the function of f_s in formulas (IV.5.22) - (IV.5.24)] they satisfy an equation

$$\frac{d^2 f_s^{n+1}}{d\xi^2} + \frac{s}{\xi}\frac{df_s^{n+1}}{d\xi} + \frac{1}{sn+2}\xi\frac{df_s}{d\xi} + \frac{s+1}{sn+2}f_s = 0 \qquad \text{(IV.5.25)}$$

and conditions

$$\int_0^\infty f_s(\xi)\xi^s d\xi = 1; \quad \left(\frac{\xi^s df^{n+1}}{d\xi}\right)_{\xi=0} = 0 \quad (s = 0,\ 1,\ 2). \qquad \text{(IV.5.26)}$$

From the continuity of gas density of ρ and flow of gas of $\rho\vec{u} = -\frac{k}{\mu}\rho\,\mathrm{grad}\,p = -\frac{k\beta^n n\,\mathrm{grad}\,\rho^{n+1}}{\mu(n+1)}$, it follows that the functions of ρ and $\mathrm{grad}\,\rho^{n+1}$ must be continuous. For one-dimensional motions this will entail the continuity of ρ and $\partial\rho^{n+1}/\partial r$, while for by us the self-similar problems in question - the continuity of $f_s(\xi)$ and $df_s^{n+1}/d\xi$.

Page 105.

After multiplying both parts of the equation (IV.5.25) to ξ^s, we will obtain on the left side of this equation total derivative. By integrating, let us find the first integral in the form:

$$\xi^s \frac{df_s^{n+1}}{d\xi} + \frac{1}{sn+2}\xi^{s+1}f_s = C_s. \qquad \text{(IV.5.27)}$$

Let us note now that the $\xi^{s+1}f_s(\xi)$ vanishes with of $\xi\to 0$, otherwise the integrals under conditions (IV.5.26) they would diverge with $\xi=0$. Therefore, by set/assuming in the equation (IV.5.27) of $\xi=0$ and by utilizing a condition (IV.5.26), we will obtain $C_0=C_1=C_2=0$. By having this in form and by integrating again of relationship/ratio (IV.5.27), easily let us find expressions for an $f_s(\xi)$ in the form:

$$f_s(\xi)=\left[\frac{n}{2(n+1)(sn+2)}(e_s-\xi^2)\right]^{1/n}; \quad 0\leqslant\xi\leqslant\sqrt{e_s};$$
$$f_s(\xi)\equiv 0 \quad (\xi\geqslant\sqrt{e_s}). \qquad \text{(IV.5.28)}$$

where c_s - postoyannaya integration. As it is not difficult to see, these solutions satisfy the formulated above requirements for the continuity of f_s and $df_s^{n+1}/d\xi$.

From the first relationship/ratio (IV.5.26) we obtain

$$\rho_0 = \begin{cases} \left[\dfrac{M_0^2}{a^2m^2t}\right]^{\frac{1}{n+2}} \left\{\dfrac{n}{2(n+1)(n+2)}\left[e_0 - r^2\left(\dfrac{m^n}{M_0^n a^2 t}\right)^{\frac{2}{n+2}}\right]\right\}^{1/n} \\ \qquad\qquad\qquad\qquad 0 \leqslant r \leqslant r_0(t); \\ 0 \quad r \geqslant r_0 = \sqrt{e_0}\left(\dfrac{M_0^n a^2 t}{m^n}\right)^{\frac{1}{n+2}}; \end{cases} \qquad \text{(IV.5.32)}$$

$$\rho_1 = \begin{cases} \left[\dfrac{M_1}{2\pi a^2 m t}\right]^{\frac{1}{n+1}} \left\{\dfrac{n}{4(n+1)^2}\left[e_1 - r^2\left(\dfrac{(2\pi m)^n}{M_1^n a^2 t}\right)^{\frac{2}{1+n}}\right]\right\}^{1/n} \\ \qquad\qquad\qquad\qquad 0 \leqslant r \leqslant r_1(t); \\ 0 \quad r \geqslant r_1(t) = \sqrt{e_1}\left(\dfrac{M_1^n a^2 t}{2\pi m)^n}\right)^{\frac{1}{1+n}}; \end{cases} \qquad \text{(IV.5.33)}$$

$$\rho_2 = \begin{cases} \left[\dfrac{M_2^{2/3}}{(4\pi m)^{2/3} a^2 t}\right]^{\frac{3}{3n+2}} \left\{\dfrac{n}{2(n+1)(3n+2)}\left[e_2 - \dfrac{r^2}{\left(\dfrac{M_2^n a^2 t}{(4\pi m)^n}\right)^{\frac{2}{3n+2}}}\right]\right\}^{1/n} \\ \qquad\qquad\qquad\qquad 0 \leqslant r \leqslant r_2(t); \\ 0 \quad r \geqslant r_2(t) = \sqrt{e_2}\left[\dfrac{M_2^n a^2 t}{(4\pi m)^n}\right]^{\frac{1}{3n+2}}. \end{cases} \qquad \text{(IV.5.34)}$$

$$e_0 = \left(\frac{2(n+1)(n+2)}{n}\right)^{\frac{2}{n+2}} \left[\frac{2\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{n}\right)}\right]^{\frac{2n}{n+2}}; \quad (IV.5.29)$$

$$e_1 = \left(2\,\frac{1+n}{n}\right)^{\frac{n}{n+1}} \left[\frac{4(n+1)^2}{n}\right]^{\frac{1}{n+1}} = [2(1+n)]^{\frac{n+2}{n+1}} n^{-1}; \quad (IV.5.30)$$

$$e_2 = \left[\frac{2(n+1)(3n+2)}{n}\right]^{\frac{2}{3n+2}} \left[\frac{2\Gamma\left(\frac{5}{2}+\frac{1}{n}\right)}{\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(\frac{3}{2}\right)}\right]^{\frac{2n}{3n+2}}, \quad (IV.5.31)$$

where Γ - Euler's gamma function. Finally we have expressions for the density distributions of gas in the form:

Page 106.

As is evident, these solutions are continuous and possess continuous derived $\partial\rho^{n+1}/\partial r$. However, derived $\partial\rho/\partial r$ has at point $r = r_s(t)$ (t) a discontinuity/interruption, so that the constructed solutions are not related to the classical solutions to equation in the partial derivatives (IV.5.18) and are its generalized solutions.

It is interesting to trace, as with tendency n to zero solutions (IV.5.32) - (IV.5.34) transfer/convert to known solutions of the type of the instantaneous source of the classical linear equation of thermal conductivity. Let us examine this in an example of solution (IV.5.32). We have with an $r \leqslant r_0(t)$

$$\rho_0 = \left[\frac{M_0^2}{a^2 m^2 t}\right]^{\frac{1}{n+2}} \left\{\left[\frac{n e_0}{2(n+1)(n+2)}\right]^{1/n}\left[1 - \frac{r^2}{r_0^2}\right]^{1/n}\right\}.$$

After using the Stirling formula for G-function at great significance of argument

$$\Gamma(z) = \sqrt{\frac{2\pi}{e}}\, e^{-z}\, [z(z+1)]^{\frac{z}{2}+\frac{1}{4}}$$

$$(z \gg 1) \quad \text{(IV.5.35)}$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$, with $n \to 0$ known expression for solution of the type of the instantaneous source of the classical linear equation of the heat conductivity:

$$\frac{M_0}{m\sqrt{a^2\pi t}}\, e^{-\frac{r^2}{4a^2t}} \quad \text{(IV.5.36)}$$

(absence of usual pair in denominator is explained by the fact that we accepted the total mass of gas of equal $2M_0$, and not M_0).

Figure IV.16 depicts the density distributions of gas at certain point in time corresponding to different values n during the identical values of all other parameters.

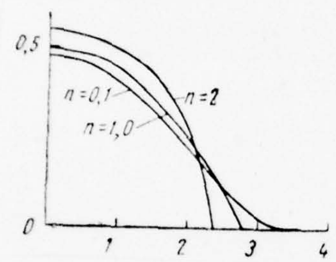


Fig. IV.16.

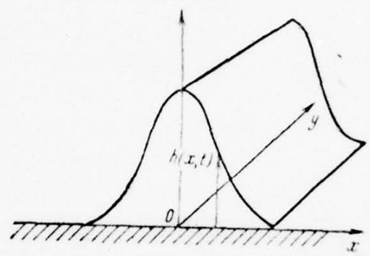


Fig. IV.17.

As can be seen from figure, during $n > 1$ curve of density distribution, it approaches the boundary of the region, occupied with gas, at right angles. With $n = 1$ this curve approaches and boundary of the region, occupied with gas, at acute angle. During $0<n < 1$, curve of the density distribution of gas approaches the boundary of the region, occupied with gas, at zero angle, concerning the axis of abscissas at the end point whose coordinate grow/rises with decrease in n. In all these cases occurs the final velocity of propagation of the leading edge of the region, occupied with gas. To limiting case $n= = 0$ (classical linear equation of thermal conductivity) corresponds, as is known, the infinite velocity of propagation of leading edge.

The motions examined above were investigated for the first time in work Ya. of B. Zeldovich and A. S. Kompaneyts [50] in connection with the mathematically analogous problem of the theory of thermal conductivity and independently - in connection with the problem of filtration theory in G. I. Barenblatt's work [5].

4. Problem of the spreading of the mound of liquid on impenetrable horizontal confining stratum. The obtained under the preceding/previous heading solutions can be used, because of the analogy between the flat free-flow motions of the incompressible

fluid and the isothermal filtration of thermodynamically perfect gas (see Chapter II), for the construction of the solution of some interesting problems of flat free-flow motions.

Let us assume that in the layer of infinite power, which lies on impenetrable confining stratum, by one method or the other is created the mound of liquid which then spreads under the action of the force on entire confining stratum (Fig. IV.17). Let us examine two simplest diagrams. In the first case let us assume that the mound of liquid is strongly elongated in one direction, so that the picture of motion can be considered not depending on the coordinate, calculated in this direction; let us designate it by y. Thus, the increase of the floating surface above confining stratum h depends only on time t against coordinates x, calculated in perpendicular direction, and it satisfies an equation

$$\frac{\partial h}{\partial t} = a^2 \frac{\partial^2 h^2}{\partial x^2}; \quad a^2 = \frac{k\rho g}{2m\mu}. \qquad \text{(IV.5.37)}$$

. Page 108.

Assuming to be, further, that the mound of liquid at the moment is concentrated in very narrow near line x = 0, we obtain the initial condition and conditions at infinity in the form h (x, 0) = 0 $(x \neq 0)$; $h(\infty, t) = 0$. Further, the total amount of liquid, per unit the unit of the width of mound in the process of motion, is constant, so that

$$\int_0^\infty h\,dx = \frac{M_0}{m}. \tag{IV.5.38}$$

This problem in accuracy is analogous to the problem examined earlier of instantaneous source during the isothermal filtration of gas by plane waves (n = 1), whereupon instead of gas density it figures as height/altitude h of floating surface. By copying solution (IV.5.32) in n = 1 in new terms, we will obtain the solution of the problem of the spreading of the concentrated flat/plane mound in the form:

$$h = \left[\frac{M_0^2}{a^2 m^2 t}\right]^{1/3} \frac{18^{1/3}}{12}\left[1 - \frac{x^2}{\left(\frac{18 M_0 a^2 t}{m}\right)^{2/3}}\right]; \tag{IV.5.39}$$

$$0 \leqslant x \leqslant x_0(t) = \left(\frac{18 M_0 a^2 t}{m}\right)^{1/3}$$

and h = 0 with $x \geqslant x_0(t)$.

In the second case let us assume that the form of the initial mound of liquid is symmetrical relative tc certain vertical axis. Then all the subsequent motion will possess symmetry relative to this axis, and the increase of floating surface will satisfy an equation

$$\frac{\partial h}{\partial t} = a^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial h^2}{\partial r}, \quad a^2 = \frac{k\rho g}{2m\mu}, \qquad \text{(IV.5.40)}$$

where r is a distance from the axis of symmetry.

If we consider that the initial mound is concentrated in small vicinity of the axis of symmetry, then the initial condition and condition at infinity take form $h(r, 0) \equiv 0,\ (r \neq 0);\ h(\infty, t) = 0$.

The condition of the constancy of an entire mass of liquid in layer M_1 takes the form:

$$2\pi m \int_0^\infty h(r,\ t)\, r\, dr = M_1. \qquad \text{(IV.5.41)}$$

This problem in turn, in accuracy is analogous to the problem of instantaneous source during the isothermal axisymmetric filtration of gas, only instead of gas density in problem is considered the

height/altitude of floating surface.

Page 109.

Set/assuming in solution (IV.5.33) n = 1 and copying it in the designations of new problem, we obtain the solution of this problem in the form:

$$h=\left[\frac{M_1}{2\pi a^2 m t}\right]^{1/2}\frac{1}{2\sqrt{2\pi}}\left[1-\frac{r^2}{8\left(\frac{M_1 a^2 t}{2\pi m}\right)^{1/2}}\right]; \qquad \text{(IV.5.42)}$$

$$0\leqslant r\leqslant r_1(t)=\sqrt{8}\left(\frac{M_1 a^2 t}{2\pi m}\right)$$

and $h\equiv 0$ with ~~при~~ $r\geqslant r_1(t)$.

5. Solution of the type of dipole. Let us examine the polytropic filtration of thermodynamically ideal gas in semi-infinite layer with flat/plane boundary. Let us assume that to start the

pressure and the density of gas in layer are negligible. At initial torque/moment into the layer through the boundary, instantly is introduced certain amount of gas after which the pressure and the density of gas on boundary they become equal to zero. Investigate the process of the spreading of gas on layer. The filtration of gas occurs by plane waves, since all the motion characteristics depend only on time t and coordinate x, calculated along the normal to the plane of the boundary of layer, by which is assigned the value of coordinate x = 0. The density of gas satisfies in this case equation

$$\frac{\partial \rho}{\partial t} = a^2 \frac{\partial^2 \rho^{n+1}}{\partial x^2}; \quad a^2 = \frac{knp_*}{\mu m (n+1) \rho_*^n}. \qquad \text{(IV.5.43)}$$

Since the introduction of gas into layer occurs, by hypothesis, instantly, and the initial density of gas is negligible, the initial condition and condition on infinity for the problem in question take the form:

$$\rho(x, 0) \equiv 0 \quad (x \neq 0); \quad \rho(\infty, t) \equiv 0. \qquad \text{(IV.5.44)}$$

After multiplying now both parts of the equation (IV.5.43) on x after integrating from x = 0 to x = ∞, we will obtain

$$\frac{d}{dt}\int_0^\infty \rho x\,dx = a^2\int_0^\infty x\,\frac{\partial^2\rho^{n+1}}{\partial x^2}\,dx = a^2\left(x\,\frac{\partial\rho^{n+1}}{\partial x}\right)\Bigg|_{x=0}^{x=\infty} - a^2\int_0^\infty \frac{\partial\rho^{n+1}}{\partial x}\,dx =$$
$$= a^2\left(x\,\frac{\partial\rho^{n+1}}{\partial x}\right)\Bigg|_{x=0}^{x=\infty} + a^2\rho^{n+1}(0,\ t) - a^2\rho^{n+1}(\infty,\ t).$$

But the expression of $(x\,\partial\rho^{n+1}/\partial x)_{x=\infty}$ is equal to zero on the strength of condition at infinity ρ (∞, t) = 0; if this value was equal to zero, then the condition of equality to zero of density at infinity would not be satisfied. Further, the value of $(x\,\partial\rho^{n+1}/\partial x)_{x=0}$ it is equal to zero, since otherwise density with x = 0 would be infinite.

Page 110.

Since it has been defined that during process gas density on boundary of x = 0 is equal to zero, the preceding/previous relationship/ratio giving

$$\frac{d}{dt}\int_0^\infty \rho x\,dx = 0;\ \int_0^\infty \rho x\,dx = Q = \mathrm{const}, \qquad \text{(IV.5.45)}$$

constant Q it characterizes in a sense the amount of gas, instantaneously entering the layer in the beginning process thus the process in question is characterized by the constancy "static torque" of density distribution.

From dimensional analysis, it is evident that the solution of the problem in question, i.e., the solution to equation (IV.5.43) under the conditions (IV.5.44) and (IV.5.45), is self-similar and is

represented in the form:

$$\rho=\left(\frac{Q}{a^2t}\right)^{\frac{1}{n+1}}f(\xi);\quad \xi=\frac{x}{(a^2Q^nt)^{\frac{1}{2(n+1)}}}. \qquad \text{(IV.5.46)}$$

Substituting expression (IV.5.46) in equation (IV.5.43) and condition (IV.5.45), we obtain for determining function $f(\xi)$ boundary-value problem

$$\frac{d^2f^{n+1}}{d\xi^2}+\frac{1}{2(n+1)}\xi\frac{df}{d\xi}+\frac{1}{n+1}f=0; \qquad \text{(IV.5.47)}$$

$$\int_0^\infty \xi f(\xi)\,d\xi=1;\quad f(0)=0,$$

whereupon as before on the strength of the necessary continuity of density and flow of gas function $f(\xi)$ it must be continuous and have continuous derived $df^{n+1}/d\xi$.

The unknown solution to this boundary-value problem takes the form:

$$f(\xi)=\begin{cases} q\left(\frac{\xi}{\xi_0}\right)^{\frac{1}{n+1}}\left[1-\left(\frac{\xi}{\xi_0}\right)^{\frac{n+2}{n+1}}\right]^{1/n}; & 0\leqslant\xi\leqslant\xi_0, \\ 0 & \xi\geqslant\xi_0, \end{cases} \quad \text{(IV.5.48)}$$

where

$$q=\left[\frac{n}{2(n+1)(n+2)}\right]^{1/n}\xi_0^{2/n},$$

a the constant of ξ_0 it is defined frcm condition (IV.5.47) analogous with that, as this was made fcr the scurces:

$$\xi_0=2^{\frac{1}{2(n+1)}}(n+2)^{1/n}(n+1)^{-\frac{n-1}{2(n+1)}}n^{-\frac{1}{2(n+1)}}\left[\mathrm{B}\left(\frac{n+1}{n},\frac{2n+1}{n+2}\right)\right]^{-\frac{n}{2(n+1)}} \tag{IV.5.49}$$

(B — B-function of Euler).

Page 111.

The coordinate of the $x^*(t)$ of the leading edge of the region, the pressure of gas in which differs from zero, changes according to relationship/ratio

$$x^*(t) = \xi_0 (a^2 Q^n t)^{\frac{1}{2(n+1)}}. \qquad \text{(IV.5.50)}$$

. To analogously preceding/previous it is possible to show that with $n \to 0$ obtained solution approaches known solution of the type of

the dipole of the classical linear equation of thermal conductivity.

Let us note that the constructed by us solution is limiting case for the solutions, examined in chapter IV, §1, and corresponding $n = 1$, $\alpha = -1/2$, $\lambda = -1$.

With $n = 1$ obtained solution can be interpreted from the viewpoint of the problem of free-flow filtration. Let on the flat/plane boundary of $x = 0$ semi-infinite layer, arrange/located on horizontal confining stratum and which does not contain liquid, suddenly can be created the very high liquid head, and then pressure head on boundary again drops to zero [1].

FOOTNOTE [1]. This problem schematically describes, for example, the filtration motion, which appears in the walls of channels after seasonal flood. ENDFOOTNOTE.

The increase of floating surface h satisfies in this case equation (IV.5.37) and condition

$$\int_0^\infty h(x, t)\, x dx = Q, \qquad \text{(IV.5.51)}$$

which is obtained from equation (IV.5.37), analogous with that, as was obtained condition (IV.5.40) from equation (IV.5.43).

The constant Q determines the amount of interstitial into layer in the beginning of process liquid. Set/assuming in relationship/ratios (IV.5.46) and (IV.5.48) n = 1 and copying these relationship/ratios in the terms of the problem of free-flow motion in question, we obtain the solution of this problem in the form:

$$h = \sqrt{\frac{Q}{a^2 t}}\, f\left[\frac{x}{(a^2 Q t)^{1/4}}\right]; \qquad \text{(IV.5.52)}$$

$$f = \begin{cases} \frac{\xi_0^2}{12}\left(\frac{\xi}{\xi_0}\right)^{1/2}\left[1-\left(\frac{\xi}{\xi_0}\right)^2\right] & 0 \leqslant \xi \leqslant \xi_0; \\ 0 & \xi \geqslant \xi_0 = 2\sqrt[4]{5}, \end{cases}$$

so that the coordinate of front/leading boundary of the region,

cccupied with liquid, it changes in the course of time according to the law

$$x^*(t) = 2\sqrt[4]{5a^2Qt} = \sqrt[4]{\frac{40k\rho gQt}{m\mu}}. \qquad \text{(IV.5.53)}$$

The coordinate of the point of the aaaaaa of the corresponding to the maximum increase floating surface, is determined by relationship/ratio

$$x^{**}(t) = \frac{2}{(1+6\sqrt[4]{5})^{3/4}}\sqrt[4]{\frac{5k\rho gQt}{2m\mu}}. \qquad \text{(IV.5.54)}$$

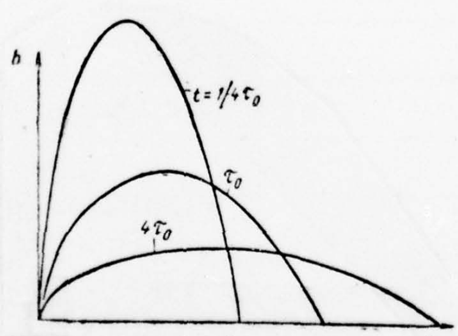


Fig. IV.18.

Figure IV.18 depicts the form of free surface of liquid, which corresponds to motion in question for several points in time (τ_0 - certain arbitrarily selected point in time). The solution examined above was obtained in G. I. Barenblatt's work and Ya. B. Zeldovich [19].

6. Self-similar motions during arbitrary equation of state. The analogy between the free-flow filtration incompressible liquid and the filtration of thermodynamically perfect gas can be propagated in the case of the filtration of gas with arbitrary pressure. During free-flow filtration this answers the frequently being encountered case of motion in the laminar soil whose properties are alternating/variable by height.

Let us examine the flat free-flow filtration motion of the incompressible fluid in the soil, permeability k and porosity m of which they depend on distance on the horizontal confining stratum z. If liquid level in this vertical section is h, then in elementary layer with a thickness dx and by width b is contained the volume of liquid

$$H^* dx = \left(\int_0^h m(z) b(z) dz \right) dx. \qquad \text{(IV.5.55)}$$

At the same time the fluid flow rate through this section composes

$$-\left(\frac{\rho g}{\mu}\int_0^h b\,(z)\,k\,(z)\,dz\right)\frac{\partial h}{\partial x}, \qquad \text{(IV.5.56)}$$

where ρ - the density of liquid.

By composing by usual method the equation of continuity, we will obtain for pressure head h differential equation

$$\frac{\partial H^*}{\partial t} = \frac{\rho g}{\mu} \frac{\partial}{\partial x} \varphi(h) \frac{\partial h}{\partial x}. \qquad \text{(IV.5.57)}$$

where

$$\varphi(h) = \int_0^h k(z)\, b(z)\, dz$$

- nondecreasing function.

Page 113.

Introducing function

$$P(h) = \int_0^h \varphi(\eta)\, d\eta, \qquad \text{(IV.5.58)}$$

easy to lead equation (IV.5.57) to the standard form:

$$\frac{\partial P}{\partial t} = \varkappa_0 K(P) \frac{\partial^2 P}{\partial x^2}, \qquad \text{(IV.5.59)}$$

where

$$\varkappa_0=\frac{\varphi(P_0)\rho g}{\mu m(P_0)b(P_0)};\quad K(P)=\frac{\varphi(P_0)m(P_0)b(P_0)}{\varphi(P_0)m(P)b(P_0)} \qquad (IV.5.60)$$

(obviously, ϕ, m and b it is possible to consider the known functions E).

The solution to equation (IV.5.59) under the conditions

$$P(0,x)=P_\infty;\quad P(t,0)=P_0 \qquad (IV.5.61)$$

is self-similar. We will examine the filtration into dry soil, for which $P_\infty=0$. The unknown solution will seem in the form:

$$P(t,x)=P_0 f(\xi);\quad \xi=\frac{x}{2\sqrt{\varkappa_0 t}}, \qquad (IV.5.62)$$

where function $f(\xi)$ it satisfies an equation

$$K(P_0 f) f''(\xi) + 2\xi f' = 0. \qquad \text{(IV.5.63)}$$

It is possible to demonstrate that if k (z) and m (z) take with small z the finite values, then as in the problem of the isothermal filtration of gas, movement for finite time covers only the limited section of layer. Therefore with $\xi \geqslant \xi_0$ $P \equiv 0$. Furthermore, from the continuity condition of expenditure/consumption it follows that $f'(\xi_0) = 0$.

We see that the problems, which differ in terms of the concrete/specific/actual form of the function m (z), b (z) and k (z), are reduced to identical the boundary/edge coefficient K. Therefore it is desirable, without solving equation (IV.5.63), to compose the concept about how is changed solution with a change in coefficient of K. Let us give a following simplest example. Let function f_1 satisfy those boundary conditions that and f, and to the equation of form

(IV.5.63), but with certain other coefficient K_1. Let us assume also that the function K (P) monotonically grow/rises and that K_1 (P) $\geq$ K (P). Then with all ξ

$$f_1(\xi) \geqslant f(\xi). \qquad \text{(IV.5.64)}$$

The proof of this affirmation is given by A. M. Pirverdyan [90], who utilized it also to evaluate certain solution with the aid of others, allow/assuming elementary expression.

Problems to §5.

1. To examine the axisymmetric self-similar solutions of the problem of Cauchy for the equation of the isothermal filtration of the thermodynamically perfect gas

$$\frac{\partial p}{\partial t} = a^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p^2}{\partial r}, \qquad \text{(IV.5.65)}$$

satisfying the initial conditions:

$$p(r, 0) = \sigma r^{\alpha};\ \sigma = \text{const} > 0;\ \alpha = \text{const} > 0$$

(G. I. Barenblatt [5, 13]).

Page 114.

Answer/response. In $0<\alpha<2$, solution of problem exists for any value of time and is expressed in the form:

$$p(r, t)=\sigma(a^2\sigma t)^{\frac{\alpha}{2-\alpha}}\mu_*^{-2}\Phi\left[\frac{\mu_* r}{\sqrt{1-\alpha/2}\,(a^2\sigma t)^{1/(2-\alpha)}}\right], \qquad \text{(IV.5.66)}$$

where f of $\Phi(\xi)$ are a solution to equation (IV.2.39) in $\lambda = \alpha/2$, that satisfies the initial conditions

$$\Phi(0)=1, \quad \Phi'(0)=0.$$

$$\mu_*=\left[\frac{C(\alpha)}{(-\alpha/2)^{\alpha/2}}\right]^{\frac{1}{2-\alpha}}, \quad C(\alpha)=\lim_{\xi\to\infty}\Phi(\xi)\,\xi^{-\alpha}, \qquad \text{(IV.5.67)}$$

In $\alpha = 2$ solution is expressed in the final form:

$$p(r, t) = \frac{\sigma r^2}{1 - 16a^2\sigma t}. \qquad \text{(IV.5.68)}$$

This solution exists only on the final interval of the values of time t: $0 \leq t \leq T$, where $T = 1/16a^2\sigma$.

With t = T, it goes to infinity simultaneously for all values of r.

In $\alpha > 2$ solution of the problem of Cauchy in question not is singular; therefore with $\alpha > 2$ this formulation of the problem turns out to be physically senseless.

2. We will consider the self-simulation of the solution of problem Cauchy for the flow of gas by the plane waves p = p (x, t) in infinite layer with are initial the conditions:

$$p(x, 0) = \sigma_1 x^{\alpha_1} \ (x > 0); \ p(x, 0) = \sigma_2 (-x)^{\alpha_2} \ (x < 0).$$

3. To examine the case of the equalization of the gradient of pressure

$$(\sigma_1 = P_1; \ \sigma_2 = P_2 \neq P_1, \ \alpha_1 = \alpha_2 = 0).$$

. By analogy between by free-flow the filtration and filtration of gas the solution of this problem at the same time describes

equalization the levels h_1 and h_2 underground water with free-flow filtration. With $h_1 > H$, $h_2 = 0$ this case answers the propagation of the initially vertical boundary between liquid and gas in layer with a power H. The problem of this motion of boundary appears during the design of underground storage of gas [34].

MT/ST-76-1860

Page 115.

Chapter V.

APPROXIMATION METHODS OF THE SOLUTION OF THE PROBLEMS OF UNSTEADY FILTRATION.

If we do not consider linear problems, obtaining the effective

exact solution of the problems of unsteady filtration will prove to be faster exception/elimination, than by rule. ; however, even, the solutions of linear problems, in theory is always available, not always satisfy the requirements for simplicity and visibility. In an even larger measure this is related to the solutions (in this book not examine/considered), obtained with the aid of electronic computers.

Diverse technical problems must be solved in a comparatively simple analytical form, which allow/assumes the qualitative analysis of solution depending on the parameters of the problem. In connection with this wide application obtained the approximate problems. The majority of these methods does not have strict substantiation, and their use is justified by the mainly fact that during the comparison of results with known exact solutions is obtained the satisfactory agreement.

Before passing to the concrete/specific/actual study of problem, let us let us point out the specific special feature/peculiarities of the problems of the unstationary filtration which in many respects causes the success of the application/use of approximation methods.

1. The majority of the problems of unsteady filtration is reduced to the parabolic equations, for which is characteristic the smoothing of disturbance/perturbations in the course of time, also, with their advance inside the region for which is examined the solution.

2. In series of problems, which are of interest for a technology, solution it has at some points of the range of motion known special feature/peculiarities, and outside the vicinities of these points the state of system are nearly unperturbed or stationary. This situation is even more important in view of the fact that the linearity of many problems is exhibited only during the considerable deflection of system from stationary state.

3. Practical interest they represent the integral characteristics of solution.

Page 116.

The noted special feature/peculiarities of the problems of unsteady filtration determine the character of approximation methods. In essence they consist in the fact that is placed at first the problem, close to datum and which has effective solution, and then search for small corrections to this solution. When the initial problem has special feature/peculiarities, is logical to pose auxiliary problem with the same special feature/peculiarities. Approximation methods are distinguished depending on which problems are utilized as "close" and how into solution are introduced supplementary corrections.

The application/use of approximation methods and to concrete/specific/actual problem has its specific character, especially in connection with the requirement for effectiveness. Therefore, without attempting to give ready formulas for each method, let us give the examples from which will be clear and the technician of the application/use of a method, and possible complications when using him in other facts.

§1. Diagram of the method integral relationship/ratio. Inflow to

gallery in infinite layer during elastic mode/conditions.

Let us examine first of all several one-dimensional problems of the theory of elastic mode/conditions. In this case,, as is known, pressure distribution is described by the linear equation of thermal conductivity

$$\frac{\partial p}{\partial t} = \varkappa \nabla^2 p, \tag{V.1.1}$$

and the use of approximation methods is connected not with the impossibility of obtaining exact solutions, but with their complexity. Furthermore, the solution of series of problems has a number systematic value.

1. Earlier (see Chapter III, §1) was already given the solution of the problem of the starting a tunnel in infinite layer, which takes the form:

$$p = p_0 - (p_0 - p_1)\,\mathrm{erf}\,\frac{x}{2\sqrt{\varkappa t}}, \tag{V.1.2}$$

where p_0 is the initial pressure in layer; p_1 - pressure on tunnel.

The difference between the initial pressure and its instantaneous value rapidly decreases.

Utilizing determination of the function of the errors in the form of integral, it is easy to show, integrating in parts that

$$|p-p_0|<|p_0-p_1|\frac{2\sqrt{\varkappa t}}{r}\exp\left(-\frac{r^2}{4\varkappa t}\right).$$

. Therefore it is logical to introduce the concept about the range of the effect of gallery, i.e. the range in which the pressure noticeably differs from its initial value. From the preceding/previous formula and from dimensional considerations it is clear that the size/dimension of the range of effect

$$l(t)=c\sqrt{\varkappa t}, \qquad (V.1.3)$$

where c - the constant of the order of one whose value it depends on as is determined the range of effect.

Page 117.

We will now search for the approximate solution of problem. Let us assume that outside the range of the effect of the gallery of motion no, and within it pressure is distributed just as with stationary filtration, i.e., it is linear. Let us accept further, that the pressure is continuous. Then

$$\begin{aligned} &p(x, t) = p_1 + (p_0 - p_1)\frac{x}{l} && [0 \leqslant x \leqslant l(t)]; \\ &p(x, t) = p_0 && [x \geqslant l(t)]. \end{aligned} \tag{V.1.4}$$

Now it suffices to determine the form of dependence l (t) in

crder complete the construction of approximate solution. The selection of relationship/ratic for determining of l (t) is to a certain extent arbitrary, since it is not possible to indicate that only relationship/ratio after satisfying which, it is possible to achieve the best conformity between the approximate solution (V.1.4) and the exact solution of stated problem. Since for the majority of application/appendices the primary meaning has the correct determination of the amount of take/selected from layer liquid, usually boundary l (t) find from the condition of material balance for a layer as a whole. For time dt past the section of the section of layer with a width of b and with a power H passes the volume of liquid bHudt, whereupon in accordance with the law of filtration

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x}\bigg|_{x=0} = \frac{k}{\mu}\frac{p_1 - p_0}{l}, \qquad \text{(V.1.5)}$$

For time t from the start past section x = 0, it passes the volume of liquid

$$Q(t) = \frac{kbH}{\mu}\int_0^t \frac{p_1 - p_0}{l(t)}\,dt. \qquad \text{(V.1.6)}$$

This volume of liquid freed (will for a certainty consider that from layer is take/selected the liquid and $p_1 < p_0$) because of a decompression in the range of motion. Assuming that the deformation of layer and liquid occurs elastic, we have

$$Q = mbH\int_0^l \frac{p-p_0}{K}\,dx = \frac{mbH}{K}\int_0^{l(t)} (p_0-p_1)\left(\frac{x}{l}-1\right)dx. \quad \text{(V.17)}$$

Equating expressions (V.16) and (V.17), we obtain

$$\frac{k}{\mu} bH (p_1-p_0)\int_0^t \frac{dt}{l(t)} = \frac{mbH}{K}(p_1-p_0)\int_0^l \left(1-\frac{x}{l}\right)dx$$

or

$$\varkappa \int_0^t \frac{dt}{l(t)} = \int_0^l \left(1 - \frac{x}{l}\right) dx = \frac{1}{2} l. \qquad (V.1.8)$$

. Page 118. The solution to equation (V.1.8), obviously, takes the form:

$$l = 2\sqrt{\varkappa t} \qquad (V.1.9)$$

(constant c in equation (V.1.3) is equal in this case of 2).

Thus, the resultant expression for a pressure will be

$$\begin{aligned} p(x, t) &= p_1 + \frac{1}{2}(p_0 - p_1)\frac{x}{\sqrt{\varkappa t}} & (0 \leqslant x \leqslant 2\sqrt{\varkappa t}); \\ p(x, t) &= p_0 & (x \geqslant 2\sqrt{\varkappa t}). \end{aligned} \qquad (V.1.10)$$

According to equation (V.1.5) the rate of filtration on the boundary of layer is changed according to the law

$$u(0,t)=\frac{k}{\mu}\frac{p_1-p_0}{2\sqrt{\varkappa t}}. \qquad (V.1.11)$$

From exact solution (V.1.2) for a rate of filtration is obtained the expression

$$u_0(0,t)=\frac{k}{\mu}\frac{p_1-p_0}{\sqrt{\pi\varkappa t}}. \qquad (V.1.12)$$

Thus, the obtained solution, giving the qualitatively accurate description of motion, is all the same only roughly approximated. Nevertheless the illustrated above method, called the method of the consecutive exchange of steady states, obtained sufficiently wide application in practical calculations. Without being stopped on further examples and the theory of elastic mode/conditions, which can be found in I. A. Charny's books [118, 119], V. N. Shchelkacheva and B. B. Lapuk [126], A. M. Pirverdyan [91] et al., let us formulate the

common diagram of method.

Is assumed that during the imposition of disturbance/perturbation entire layer distinctly divide/marked off into two range - the range of the undisturbed state and the range of disturbance/perturbation and that pressure in the range of disturbance/perturbation distributed in the manner that if motion in this range was stationary, and outside the range of disturbance/perturbation - disturbance/perturbation was completely absent. Finally, with the aid of certain supplementary condition as which most frequently is selected the equation of material balance, is determined the law of the growth of the range of disturbance/perturbation.

Thus, the method of the consecutive exchange of steady states is based on three assumptions: 1) there is a finite domain of the disturbed motion; 2) motion within this domain stationary; 3) the size/dimension of the domain of the disturbance/perturbation of is determined from the condition of material balance.

The first of these three assumptions by itself will not

introduce considerable error, since, as is known from examples, the disturbed motion very rapidly attenuates with removal/distance from the place of disturbance/perturbation. The second assumption is connected with the fact that the motion of liquid near perturbation source comparatively rapidly stabilizes.

Page 119.

However, here arbitrary is the assumption about that which stabilization occurs immediately in an entire range of possible motion. The third assumption is completely logical, although the selection of precisely this supplementary condition is not completely necessary.

2. The given reasonings show that the possible refinements of the method of the consecutive exchange of steady states are connected first of all with the replacement of the second assumption by different, more corresponding the true position of businesses, and with the appropriate change in those supplementary conditions, on the basis of which is determined the size/dimension of the range of disturbance/perturbation.

The consecutive carrying-out of this program leads to the method of "integral relationship/ratios", which is the standardized method which makes it possible to obtain the solution of the series of problems of unsteady filtration with sufficient practical accuracy.

Let p (x, t) are a solution to equation (V.11) in the case of the rectilinear-parallel motion:

$$\frac{\partial p}{\partial t} = \varkappa \frac{\partial^2 p}{\partial x^2}. \qquad (V.1.13)$$

After multiplying this equation for the arbitrary function f (x, t) and after integrating over x within limits from L_1 (t) to L_2 (t), we will obtain equation

$$\int_{L_1}^{L_2} \frac{\partial p}{\partial t} f(x, t)\, dx = \varkappa \int_{L_1}^{L_2} \frac{\partial^2 p}{\partial x^2} f(x, t)\, dx, \qquad (V.1.14)$$

valid with any L_1, L_2, f (x, t) and t.

Let there be in this case the family of such functions of f_n (x, t) (n = 0.1 ...), that with any t this family is full (as family of functions of x) on cut [L_1 (t), L_2 (t)]. Let, further, the relationship/ratio (V.1.14) be fulfilled for all f_n. Then, if the derivatives $\partial p/\partial t$ and $\partial^2 p/\partial x^2$ are continuous, then is function p (x, t) satisfies an equation (V.1.13) with L_1 (t) $\leq x \leq L_2$ (t). Thus, between the system of the integral equalities (V.1.14), written for a complete system of functions, and the differential equation (V.1.13) there is a equivalency, and instead of the solution to equation (V.1.13) it is possible to search for the solution of the system of equations of form (V.1.14).

Let us take the simplest complete system of functions - the consecutive degrees of three-dimensional/space variable

$$1, x, x^2, \ldots, x^n, \ldots$$

From (V.1.14) we have

$$\int_{L_1(t)}^{L_2(t)} x^n \frac{\partial p}{\partial t}\, dx = \varkappa \int_{L_1(t)}^{L_2(t)} x^n \frac{\partial^2 p}{\partial x^2}\, dx \qquad (n = 0, 1, \ldots). \quad \text{(V.1.15)}$$

Page 120.

The left side of this equation can be presented, by utilizing a formula of differentiation of definite integral, in the form:

$$\int_{L_1(t)}^{L_2(t)} x^n \frac{\partial p}{\partial t} dx = \frac{d}{dt} \int_{L_1(t)}^{L_2(t)} p x^n dx - p(L_2, t) L_2^n \frac{dL_2}{dt} + p(L_1, t) L_1^n \frac{dL_1}{dt}.$$

In right side it is possible to conduct integrations in parts. As a result after simple calculations, we will obtain from (V.1.15)

$$\frac{d}{dt} \int_{L_1(t)}^{L_2(t)} p(x, t) x^n dx = \varkappa L_2^n \left(\frac{\partial p}{\partial x}\right)_{x=L_2} - \varkappa L_1^n \left(\frac{\partial p}{\partial x}\right)_{x=L_1} - \varkappa n L_2^{n-1} p(L_2, t) + \varkappa n L_1^{n-1} p(L_1, t) + n(n-1)\varkappa \int_{L_1}^{L_2} p(x, t) x^{n-2} dx + p(L_2, t) L_2^n \frac{dL_2}{dt} - p(L_1, t) L_1^n \frac{dL_1}{dt}. \quad \text{(V.1.16)}$$

During the construction of the approximate solution to conveniently use the system of integral relationship/ratios (V.1.6), than by the initial differential equation, since in expression (V 1.16) do not enter the derivatives of unknown functions.

3. Let us examine again the problem of the disturbance of initially steady motion in layer. Let at torque/moment t = 0 pressure

$$p(x,\ 0)=P+Gx, \qquad \text{(V.1.17)}$$

be is distributed according to the law

to the answering selection of liquid from layer with expenditure/consumption - $kbHG/\mu$ (specifically, G = 0 it corresponds to the absence of motion in the undisturbed layer). Let, further, the disturbance/perturbation appear as a result of certain change in the boundary conditions of x = 0. Then, obviously, at each torque/moment pressure change in the moved away points of layer is small. Therefore, it is natural, with the search of the approximate solution to again introduce the concept about the finite domain of effect $0 \leq x \leq l(t)$, assuming to be that on boundary of $x = l(t)$ the pressure and fluid flow rate did not have time to change and retain initial value (in the case of the final layer with an extent of L the domain of effect, beginning with certain torque/moment t^*, covers layer wholly, and $l(t) = L$, $t > t^*$).

Page 121.

We will search for the approximate solution of problem in the form of polynomial

$$p(x,\ t) = P_0(t) + P_1(t)\,x/l + \ldots + P_n(t)\,x^n/l^n \quad (0 \leqslant x \leqslant l);$$
$$p(x,\ t) = p(x,\ 0) \quad (x \geqslant l). \qquad \text{(V.1.18)}$$

The expression (V.1.18) completely is determined by n + by 2 unknown functions of time - by coefficients $P_0, P_1, \ldots, P_n$ and by the position of the boundary of the region of effect l.

For determining these n + 2 unknowns it is possible to compose the system of equations, which includes certain number of integral relationship/ratios (V.1.16), the boundary condition with x = 0, determined by the setting of problem, and conditions with x = l.

The first of these conditions is continuity of pressure

$$p(l, t) = P + Gl. \qquad \text{(V.1.19)}$$

Analogously the continuity condition of expenditure/consumption gives

$$\frac{\partial p(l, t)}{\partial x} = G. \qquad \text{(V.1.20)}$$

AD-A044 774 FOREIGN TECHNOLOGY DIV WRIGHT-PATTERSON AFB OHIO F/G 13/11
THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS (TEORIY--ETC(U)
JAN 77 G I BARENBLATT, V M YENTOV, V M RYZHIK
UNCLASSIFIED FTD-ID(RS)T-1860-76-PT-1 NL

5 OF 5
AD
A044774

END
DATE
FILMED
10-77
DDC

Finally, this system of conditions can be supplemented by the conditions of the determined smoothness of solutions on the boundary of the disturbed domain of

$$\frac{\partial^2 p(l,t)}{\partial x^2} = \ldots = \frac{\partial^k p(l,t)}{\partial x^k} = 0. \qquad \text{(V.1.21)}$$

The basic question during the application/use of a method of integral relationship/ratios lies in the fact that, which from the infinite number of conditicns one should utilize for determining unknowns. It is clear that it is necessary to utilize at least one of the integral relationship/ratics, since otherwise in no way will be used equation (V.1.13). Compulsorily also must be used boundary condition with x = 0, since it reflects the specific character of problem. For those considerations must be accepted the continuity condition of pressure and expenditure/consumption with x = l. Considerably to complexly give any recommendations by choice of the remaining determining relationship/ratics. Let us note only that each integral relationship/ratio adds one differential equation, but each condition of smoothness with x = l, is one finite relation.

At the same time the conditions when x = 1, while local, they can not ensure a good approximation of solution in the basic domain. Moreover, acceptance of too large a number of such conditions can lead to the qualitative distortion of solution - to the appearance of oscillation/vibrations etc. From this viewpoint, the use of integral relationship/ratios as the supplementary determining conditions during an increase in the order of approximation - is more justified, although is more complicated.

4. Let us use these common/general/total considerations to the formulated above problem of the launching/starting of gallery. In this case

$$p(0, t) = p_1; \qquad p(x, 0) = p_0 = 0 \qquad \text{(V.1.22)}$$

(it is convenient to accept the initial value of pressure for zero).

Pressure distribution we will search for in the form (V.1.18). Utilizing a condition (V.1.22), we obtain

$$P_0(t) = p_1, \quad P_0 + P_1 + \ldots + P_n = 0. \qquad \text{(V.1.23)}$$

. Page 122.

Set/assuming in (V.1.16) l_1 (t) $\equiv$ 0, l_2 (t) = l (t), we find integral relationship/ratios in the form:

$$\frac{d}{dt}\int_0^l p\,dx = -\varkappa\left(\frac{\partial p}{\partial x}\right)_{x=0}; \qquad \text{(V.1.24)}$$

$$\frac{d}{dt}\int_0^l px\,dx = \varkappa p(0,\,t); \qquad \text{(V.1.25)}$$

$$\frac{d}{dt}\int_0^{l(t)} px^k\,dx = \varkappa k(k-1)\int_0^l p(x,\,t)\,x^{k-2}\,dx \quad (k\geqslant 2). \qquad \text{(V.1.26)}$$

. If we select n = 1 and to utilize as the only missing condition an integral relationship/ratio (V.1.24), then we will return to solution by the method of the consecutive exchange of stationary state. We will now more precisely formulate solution, using approach/approximation by the polynomials of higher order. We will set n = 2 and add one additional condition.

Let us take first as this supplementary condition

$$\frac{\partial p(l, t)}{\partial x}\bigg|_{x=l-0} = 0.$$

Then

$$P_0 = p_1; \quad P_0 + P_1 + P_2 = 0; \quad P_1 + 2P_2 = 0, \qquad (V.1.27)$$

a from (V.1.24) follows relationship/ratio

$$\frac{d}{dt}\left[p_1 l + \frac{1}{2} P_1 l + \frac{1}{3} P_2 l\right] = -\varkappa \frac{P_1}{l}.$$

Whence we find

$$P_0 = p_1; \quad P_1 = -2p_1; \quad P_2 = p_1; \quad l^2 = 12\varkappa t. \qquad (V.1.28)$$

. Respectively for a rate of filtration on boundary of u (0, t) we obtain

$$u(0, t) = -\frac{k}{\mu}\frac{P_1}{\sqrt{3\varkappa t}}, \qquad \text{(V.1.29)}$$

that already very closely to exact expression (V.1.12). Let us look now, which will be obtained, if we as supplementary condition utilize the second integral relationship/ratio (V.1.25).

We have a system of equations

$$\left.\begin{aligned} &P_0 = p_1; \quad P_0 + P_1 + P_2 = 0; \\ &\frac{d}{dt}\left[p_1 l + \frac{1}{2} P_1 l + \frac{1}{3} P_2 l\right] = -\frac{\varkappa P_1}{l}; \\ &\frac{d}{dt}\left[\frac{1}{2} p_1 l^2 + \frac{1}{3} P_1 l^2 + \frac{1}{4} P_2 l^2\right] = \varkappa p_1. \end{aligned}\right\} \qquad \text{(V.1.30)}$$

. Page 123.

Integration of the last/latter relationship/ratio gives

$$l^2(6p_1+4P_1+3P_2)=12\varkappa p_1 t.$$

Expressing P_2 through P_1, it is obvious, that

$$\left.\begin{aligned} &l^2(3p_1+P_1)=12\varkappa p_1 t;\\ &\frac{d}{dt}\left[l(4p_1+P_1)\right]=-\frac{6\varkappa P_1}{l}.\end{aligned}\right\} \qquad (V.1.31)$$

In the given case it is evident that

$$l=c\sqrt{\varkappa t};\qquad c=\text{const}, \qquad (V.1.32)$$

so that the solution of problem is simplified. From (V.1.32) it follows

$$P_1=\frac{12p_1}{c}-3p_1=\text{const}$$

and

$$l^2 = -\frac{12\varkappa t P_1}{4p_1 + P_1}; \quad c^2 = -\frac{12P_1}{4p_1 + P_1}; \quad c^2 = \frac{12p_1}{3p_1 + P_1};$$

$$\frac{P_1}{p_1} = -\frac{4p_1 + P_1}{3p_1 + P_1}; \quad P_1^2 + 4p_1P_1 + 4p_1^2 = 0; \quad P_1 = -2p_1.$$

Consequently, the solution, found thus, coincides with solution (V.1.28) - (V.1.29).

Let us examine now, which can give the following approach/approximation (n = 3).

According to common diagram we have solution in the form:

$$p(x, t) = P_0 + P_1 x/l + P_2 x^2/l^2 + P_3 x^3/l^3; \quad x < l.$$

. If we accept as supplementary conditions

$$p(0,\ t)=p_1;\quad p(l,\ t)=0;\quad \partial p(l,\ t)/\partial x=0$$

and the first two integral relationship/ratios (V.1.24), then we will obtain system of equations

$$P_0=p_1;\quad P_0+P_1+P_2+P_3=0;\quad P_1+2P_2+3P_3=0;$$
$$\frac{d}{dt}\left[p_1l+\frac{1}{2}P_1l+\frac{1}{3}P_2l+\frac{1}{4}P_3l\right]=\varkappa\frac{P_1}{l};$$
$$\frac{d}{dt}\left[\frac{1}{2}p_1l^2+\frac{1}{3}P_1l^2+\frac{1}{4}P_2l^2+\frac{1}{5}P_3l^2\right]=\varkappa p_1.$$

The solution of this system by us actually is already found. It, apparently, is given by expressions

$$P_0 = p_1;\quad P_1 = -2p_1;\quad P_2 = p_1;\quad P_3 = 0;\quad l^2 = 12\varkappa t,$$

since it satisfies system (V.1.27) — (V.1.27a), and to system (V.1.30).

Page 124.

Thus, the obtained third approach/approximation coincides with the second.

Let us select now another system of the determining conditions.

Let us require the execution conditions

$$p(0, t) = p_1;\quad p(l, t) = 0$$

and three of first integral relationship/ratios. Then for determining P_1, P_2, P_3 and P_0 we have the following system:

$$P_0 = p_1; \quad P_0 + P_1 + P_2 + P_3 = 0;$$

$$\frac{d}{dt}\left[P_0 l + \frac{1}{2} P_1 l + \frac{1}{3} P_2 l + \frac{1}{4} P_3 l\right] = -\varkappa \frac{P_1}{l};$$

$$\frac{d}{dt}\left[\frac{1}{2} P_0 l^2 + \frac{1}{3} P_1 l^2 + \frac{1}{4} P_2 l^2 + \frac{1}{5} P_3 l^2\right] = \varkappa p_1;$$

$$\frac{d}{dt}\left[\frac{1}{3} P_0 l^3 + \frac{1}{4} P_1 l^3 + \frac{1}{5} P_2 l^3 + \frac{1}{6} P_3 l^3\right] =$$

$$= 2\varkappa\left[P_0 l + \frac{1}{2} P_1 l + \frac{1}{3} P_2 l + \frac{1}{4} P_3 l\right]. \qquad \text{(V.1.33)}$$

. And in this case the solution is facilitated by the fact that from the dimensional considerations of $l = c\sqrt{\varkappa t}$, a all p_i can be only constants. Therefore equations (V.1.33) are reduced to algebraic system

/

$$P_0 = p_1; \quad P_0 + P_1 + P_2 + P_3 = 0;$$

$$c^2\left[P_0 + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{1}{4}P_3\right] = -2P_1;$$

$$c^2\left[\frac{1}{2}P_0 + \frac{1}{3}P_1 + \frac{1}{4}P_2 + \frac{1}{5}P_3\right] = P_0;$$

$$c^2\left[\frac{1}{3}P_0 + \frac{1}{4}P_1 + \frac{1}{5}P_2 + \frac{1}{6}P_3\right] = \frac{4}{3}\left[P_0 + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{1}{4}P_3\right].$$

By eliminating the unknowns P_0, P_1, P_2 and P_3, we come to the following cubic equation for c^2:

$$c^6 - 84c^4 + 1440c^2 - 9600 = 0,$$

only real root of which

$$c^2 \approx 63{,}78.$$

For remaining unknowns we have

$$P_0 = p_1;\quad P_1 = -4{,}67p_1;\quad P_2 = 6{,}79p_1;\quad P_3 = -3{,}12p_1.$$

For a rate of filtration on boundary

$$u(0,\ t) = \frac{k_1 P_1}{\mu l} = -\frac{k_1}{\mu}\frac{0{,}583p_1}{\sqrt{\varkappa t}} = -\frac{k_1}{\mu}\frac{p_1}{\sqrt{2{,}93\varkappa t}}.$$

. Page 125.

Compare the now first three approach/approximations: the first

$$p(x,\ t) = p_1\left(1 - \frac{x}{2\sqrt{\varkappa t}}\right)\quad (x \leqslant 2\sqrt{\varkappa t});$$

second

$$p(x,t) = p_1\left(1 - \frac{x}{3\sqrt{\varkappa t}} + \frac{x^2}{12\varkappa t}\right) \quad (x \leqslant \sqrt{12\varkappa t});$$

third

$$p(x,t) = p_1\left(1 - 0{,}583\,\frac{\varkappa}{\sqrt{\varkappa t}} + 0{,}107\,\frac{x^2}{\varkappa t} + 0{,}0061\,\frac{x^3}{(\varkappa t)^{3/2}}\right)$$
$$(x \leqslant 7{,}98\sqrt{\varkappa t}).$$

The results of the calculation for three approach/approximations are shown in Fig. V.1 together with exact solution (numerals of curves correspond to the number of

approach/approximation, zero answers exact solution).

From the given example is clear the diagram of the application/use of a method of integral relationship/ratios to the problems of elastic mode/conditions. It is clear also, that the construction of approach/approximations by the polynomials of the high order is encountered the difficulty not only of computational, but also fundamental character. First of all there are no any substantiated rules for the selection of that or different of several possible supplementary conditions. The second difficulty is connected with the fact that the approach/approximation by polynomials can give the solutions of the physically inadmissible form (for example negative on certain section, see the average/mean curve in Fig. V.1) during the attempt to raise the accuracy of approach/approximation.

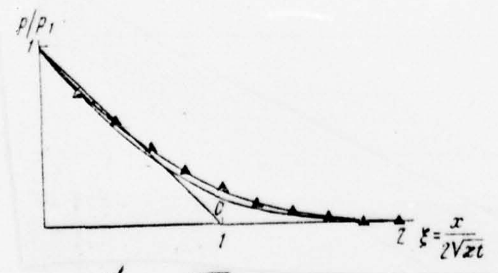


Fig. V. 1.

MT/ST-76-1860

Pages 126-154.

§2. Solution of the problems of elastic mode/conditions by the method of integral correlations.

Let us give still several examples of the use of a method of integral correlations for the solution of the problem of elastic mode/conditions. From the comparison of the obtained solutions with the appropriate "precise" solutions, are obvious the advantages of

clarity and visibility, reached with the aid of the method of integral correlations.

1. Axisymmetrical task of the launching/starting of bore hole in infinite layer. Let us examine one additional self-similar problem - the launching/starting of the bore hole of a zero radius in infinite layer.

Let us derive first those integral correlations by which must satisfy pressure distribution in axisymmetrical task.

From the fundamental equation of the distribution of pressure

$$\frac{\partial p}{\partial t}=\varkappa\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) \qquad \text{(V.2.1)}$$

after multiplication by r^{K+1} and integration within limits from R_1 to R_2, we will obtain, by analogy with correlation (V1.16), the identities: with $k = 0$ (equation of material balance)

$$\frac{d}{dt}\int_{R_1}^{R_2} p(r, t)\, r\, dr = \varkappa R_2\left(\frac{\partial p}{\partial r}\right)_{r=R_2} - \varkappa R_1\left(\frac{\partial p}{\partial r}\right)_{r=R_1} + p(R_2, t)\, R_2\frac{dR_2}{dt} - p(R_1, t)\, R_1\frac{dR_1}{dt};$$

with k > 0

$$\frac{d}{dt}\int_{R_1}^{R_2} p(r, t)\, r^{k+1}\, dr = \varkappa R_2^{k+1}\left(\frac{\partial p}{\partial r}\right)_{r=R_2} - \varkappa R_1^{k+1}\left(\frac{\partial p}{\partial r}\right)_{r=R_1} - \varkappa k R_2^k p(R_2, t) + \varkappa k R_1^k p(R_1, t) + \varkappa k^2\int_{R_1}^{R_2} p(r, t)\, r^{k-1}\, dr + p(R_2, t)\, R_2^{k+1}\frac{dR_2}{dt} - p(R_1, t)\, R_1^{k+1}\frac{dR_1}{dt}. \quad (V.2.2)$$

We will use these correlations in order to obtain the approximation solution to the task of the launching/starting of bore hole - the basic task for the rumerous methcds of the study of bore holes.

Let us accept the initial (constant) pressure in layer for zero. We will consider that at torque/moment t = 0 it begins the ring-off

of liquid from the layer through the bore hole of a negligible radius. Assuming that the selection occurs in a constant rate, we have the additional conditions:

$$p(r, 0)=0; \quad \lim_{r \to 0}\left(r \frac{\partial p(r, t)}{\partial r}\right)=\frac{\mu Q}{2 \pi k h}=q. \qquad (V.2.3)$$

precise the solution to this task as it was shown into 2, chapter III, takes the form:

$$p(r, t)=\frac{q}{2} \operatorname{Ei}\left(-\frac{r^2}{4 \varkappa t}\right). \qquad (V.2.4)$$

2. The approximate solution of task. Let us introduce newly

increasing in time radius l (t) and let us assume that with r > l (t)

$$p(r,\, t) = 0.$$

. Page 127.

In this case, the integral correlations (V.2.2), written down for cut 0 < r < l (t), take the form:

$$\frac{d}{dt}\int_0^{l(t)} rp(r,\, t)\, dr = -\varkappa \lim_{r\to 0}\left(r\frac{\partial p}{\partial r}\right) = -\varkappa q; \qquad (V.2.5)$$

$$\frac{d}{dt}\int_0^{l(t)} r^{k+1} p(r,\, t)\, dr = \varkappa k^2 \int_0^{l} p(r,\, t)\, r^{k-1}\, dr \quad (k \geqslant 1). \qquad (V.2.6)$$

. As it follows from boundary condition with $r \to 0$ [second condition (V.2.3)], the unknown solution possesses with $r \to 0$ with the peculiarity that at $\partial p/\partial r \approx q/r$. Therefore the drawing near function let us select so that it would have the same special feature/peculiarity, i.e., let us accept

$$p(r, t) = q \ln \frac{r}{l} + P_0 + P_1 \frac{r}{l} + \ldots + P_n \frac{r^n}{l^n}. \qquad (V.2.7)$$

Just as during plane-parallel motion, rough approximation is obtained under the assumption that

$$P_1 = P_2 = \ldots = P_n = 0.$$

From the continuity condition of pressure with r = l, we have also P_0 = 0; therefore remains only one unknown function l (t) which is determined with the aid of one integral correlation. As this correlation let us take the equation of material balance (V.2.5). Simple calculation gives

$$l^2 = 4\varkappa t, \qquad (V.2.8)$$

so that in zero approximaticn

$$p_0(r, t) = q \ln \frac{r}{2\sqrt{\varkappa t}} \quad (r \leqslant 2\sqrt{\varkappa t});$$
$$p_0(r, t) = 0 \quad (r \geqslant 2\sqrt{\varkappa t}). \qquad (V.2.9)$$

Zero approximaticn, cbtained in this manner, again coincides with sclution by the method of the sequential change of the staticnary states (recall that the pressure in stationary flat/plane-radial flow linearly depends on ln r).

With the searching of higher approximaticns for determining unknowns, are necessary additional conditiona. As them it is possible to use either the subsequent integral ccrrelaticns, which answer k

$\neq$ 0 or additional conditions for derivatives of pressure in terms of a radius. Actually, just as in plane-parallel flow, the system of the determining conditions it is possible tc supplement by conditions coupling/joining

$$\frac{\partial^k p(l, t)}{\partial r^k} = 0 \quad (k = 2, \ldots). \qquad \text{(V.2.10)}$$

Fig. V.2.

Therefore for determining unknown coefficients in formula (V.2.7) along with integral correlations it is possible to use expressions (V.2.10).

Let us be limited to the first approximation:

$$p(r, t) = q \ln \frac{r}{l(t)} + P_0(t) + P_1(t) \frac{r}{l(t)} \qquad \text{(V.2.11)}$$

we will determine unknowns so that would be fulfilled integral correlation (V.2.5) and conditions $p(l, t) = \partial p(l, t)/\partial r = 0.$ t)

Then $P_0 + P_1 = 0; \qquad q = P_1;$

$$\frac{d}{dt}\left[-\frac{ql^2}{4} + \frac{P_0 l^2}{2} + \frac{P_1 l^2}{3}\right] = -\varkappa q,$$

whence

$$l=\sqrt{12\varkappa t} \quad \text{and} \quad p(r, t) =$$
$$= q \ln \frac{r}{\sqrt{12\varkappa t}} - q + q\frac{r}{\sqrt{12\varkappa t}}. \tag{V.2.12}$$

Figure V.2 gives the comparison of exact solution (V.2.4) -

point - with two approximations of solution: (V/2/9) - broken curve and (V.2.12) - unbroken curve. As is evident, already the first approximation ensures sufficiently high accuracy.

3. The dignity of the method of integral correlations even clearer comes out during the solution of the problems of unsteady motion in the limited layer when it is not possible to disregard the influence of boundaries. Of course, and for these tasks it is possible without special difficulties to write solutions, by using the usual methods of mathematical physics. However, these solutions are represented in the form of Fourier series (plane-parallel motion) or of Fourier-Bessel (flat/plane-radial motion) and that is why are difficult visible. Difficulties are aggravated by the fact that even the simplest monotonous solutions are decompose/expanded according to oscillating functions, and for obtaining good approach/approximation it is necessary to undertake a large number of terms of a series.

During the application/use of a method of integral relationship to the limited layer, the investigated time interval is divide/marked off into two parts. In extent/elongation the first of them occurs the disturbance propagation (for example the region, included by motion) from that place where it arose, to the boundaries of layer. In this

case, those boundaries which the disturbance/perturbation still not reached, do not render effect on solution, so, in the launching/starting of the gallery, arrange/located at certain distance L from the impermeable boundary of layer, approximate solution will be in no way distinguished from the appropriate solution for the unlimited layer, thus far l (t) < L.

Page 129.

It is accepted to call the time interval, during which does not show the influence of boundaries, by the first phase of filtration. By the second phase of filtration, it is understood motion beginning with that torque/moment when the boundary of the region of influence reaches the moved away boundary of layer, and solution begins to depend on conditions on this boundary. It is natural that this separation into phases conditionally, and the duration of the first phase depends substantially on which approximate solution is used. So, during the solution of the problem mentioned above by the method of the sequential change of steady states (the first approximation of the method of integral correlations) $l = 2\sqrt{\varkappa t}$ and duration of first phase $t_1 = L^2/4\varkappa$. At the same time in the third approach/approximation of $l \approx 8\sqrt{\varkappa t}$ and $t_1 \approx L^2/64\varkappa$. However,

this difference insignificantly shows up in pressure distribution.

Let us be limited here only to one example, sufficiently well illustrating the possibilities of method.

Let us examine the circular layer, on outline of which ($r = R$) is supported the constant pressure, equal to the initial pressure in layer. This pressure we will as before assume/take as zero. At the moment is produced the launching/starting of the bore hole of a negligible radius, arrange/located in the center of layer. Fluid flow rate, take/selected from bore hole, is considered as before constant. Then up to the torque/moment of $t = t_1 = R_1^2/12\varkappa t$ for the distribution of equation is correct, in the first approximation,, relationship (V.2.12). With $t > t_1$, it is necessary to consider boundary condition of the nourishment:

$$p(R,\ t) = 0. \qquad \text{(V.2.13)}$$

Let us accept further, that with $t > t_1$ $l(t) \equiv R^1$).

FOOTNOTE [1]. Frequently are encountered the "substantiations" of this assumption, connected with one or the other physical interpretation of a "radius of the influence" of ℓ (t). This interpretation is completely optional. With the same success it would be possible and further to use the idea (V.2.11) with ℓ > R, according to to examine solution only in r < R, to record/write integral correlations only for this section and to consider additional conditions in r = R. ENDFOOTNOTE.

Then idea (V.2.11) will take the form:

$$p(r,\ t) = q \ln \frac{r}{R} + P_0(t) + P_1(t) \frac{r}{R}, \qquad \text{(V.2.14)}$$

whereupon from condition p (R, t) = 0 it follows $P_0 = P_1$.

For determining the only remaining unknown function P_0 (t) we

will use the first integral correlation (V.2.2). Set/assuming here $R_1 = 0$, $R_2 = R$ and taking into account (V.2.14), we will obtain

$$\frac{d}{dt}\int_0^R p(r,\ t)\, r\, dr = \frac{1}{6} R^2 \frac{dP_0}{dt} = -\varkappa P_0.$$

. Page 130.

The obtained differential equation for P_0 must be solved under condition $P_0|_{t=t_1} = -q$, which follows from the requirement for the continuity of pressure with $t = t_1$ and of correlations (V.2.11) and (V.2.14). The corresponding solution takes the form:

$$P_0(t) = -q \exp\left[-\frac{6\varkappa (t-t_1)}{R^2}\right]. \qquad (V:2.15)$$

Thus approximation for pressure distribution will be

$$p(r,t) = q \ln \frac{r}{R} - q\left(1 - \frac{r}{R}\right) \exp\left[-\frac{6\varkappa (t-t_1)}{R^2}\right]. \qquad (V.2.16)$$

As is evident, pressure distribution rapidly approaches stationary.

4. Before passing to more complex problems, let us examine a question concerning how, without having exact solution, to estimate degree of approximation, reached with the aid of the method of integral correlations. Difficulty here entails the fact that there is no criterion, which makes it possible to determine previously, how much it is necessary to take approach/approximations in order to obtain solution with the assigned accuracy. Moreover, only in exclusively rare cases can be determined, how constructed a solution is distinguished from the precise. In this case, just as in many other problems, connected with the searching of the effective approximate solution, usually are used two criterion for the accuracy of the approximate solution: the first is checking on the close in setting problems, which allow/assume exact solution (as this was made above); the second - the solution of problem with sequential to the increase in the number of terms of the drawing near polynomial. The calculation is conducted until a difference in two approximate solutions becomes the less rated value. As concerns practical calculations, in them almost always they are limited to three members in approximation for a pressure.

The examined in the last two paragraphs method of the integral relationships was proposed for the solution of the nonstationary problems of filtration theory by G. I. Barenblatt [10] and repeatedly

used by a series of the researchers. , until now, widely is used also the rougher, but simpler method of the sequential change of steady states [120] and its change in form given by A. M. Pirverdyan [90].

Together with the examined methods of the sequential replacement of the stationary states and integral correlations frequently is used also the method of the averaging of time derivative in the appropriate equation. This method is analogous to the method of Slezkin-Targa [105] in boundary-layer theory; into the hydrodynamic theory of filtration it is introduced by the works of S. D. Sokolov [108] and of G. P. Guseynova [39].

Just as in examined above methods, entire layer is divide/marked off into the range of motion (excited range) and the range of rest; in the range of motion, derivative of time is substituted by its by the average on range of values.

Page 131.

After this the pressure distribution in the zone of motion is

determined means to the solution of the ordinary differential equation of the second order. The solution contains as the parameters the average value of time/temporary derivative and the extent of the zone of motion. For their determination by a usual method are used the boundary conditions of coupling/joining and integral correlations. In such a manner as in boundary-layer theory, this method is the version of the method of integral correlations.

§3. Solution of the problems of the nonstationary filtration of gas.

1. For the problems of the filtration of gas, just as for the close to them problems of filtration in the inelastic-deformed medium, methods of approzimation compose practically the only means for an effective analytical study, if we do not consider the not numerous self-simulating cases.

The quite wide field of application it has a method, indicated already as L. S. Leybenzon [71]. this method entails the fact that instead of the nonlinear differential equation

$$\frac{\partial p}{\partial t}=\frac{k}{2m\mu}\left(\frac{\partial^2 p^2}{\partial x^2}+\frac{\partial^2 p^2}{\partial y^2}+\frac{\partial^2 p^2}{\partial z^2}\right) \qquad (V.3.1)$$

is examined linear (relative to p^2) equation

$$\frac{\partial p^2}{\partial t}=\frac{kp_0}{m\mu}\left(\frac{\partial^2 p^2}{\partial x^2}+\frac{\partial^2 p^2}{\partial y^2}+\frac{\partial^2 p^2}{\partial z^2}\right), \qquad (V.3.2)$$

where p_0 is certain constant pressure. It is obvious, this equation it is obtained from expression (V.3.1), if we multiply this equation by p, and then to replace in the coefficient before bracket p by p_0. As p_0 usually is undertaken certain characteristic pressure. L. S. Leybenzon introduced for the first time such transformation in connection with the problem of pressure change in the initially undisturbed layer, and hearth p_0 it it understood pressure in the undisturbed part of the layer. This method of information of nonlinear equation (V.3.1). to linear (V.3.2) is called linearization according to L. S. Leybenzon.

To linear equation (V.3.2) let us use entire detailed apparatus of the theory of thermal conductivity (and the theory of elastic mode/conditions). The most important dignity of method lies in the fact that it has very wide field of application - both during the solution of one-dimensional and multidimensional problems, under any law of change in boundary values of pressure and fluid flow rate etc. This determined the wide application of a linearization method in the theory of the development of gas fields. However, this method has the deficiency/lack: during its application/use the specific character of problem, which distinguishes it from the problems of elastic mode/conditions, correctly it is considered only in those fields where the motion can be considered stationary [actually, in such fields $\partial p/\partial t = \partial p^2/\partial t = 0$ and equations (V.3.1) and (V.3.2) they coincide].

Page 132.

At the present time there is an already sufficiently significant experience/experiment of the application/use of L. S. Leybenzon's

method. It turned out to be very effective during the solution to the problems in which initially fixed gas begins to move under the effect of local disturbance/perturbations. Typical in this respect is the problem of the launching/starting of gas well in infinite layer.

After linearization it is possible to use prepared/finished solution from chapter III, 2. We have

$$p^2 - p_0^2 = \frac{q\mu p_0}{2\pi k}\,\mathrm{Ei}\left(-\frac{r^2 m\mu}{2kp_0}\right), \qquad \text{(V.3.3)}$$

where r - a distance of point from the axis of bore hole; p_0 is the initial pressure in layer, and q - the volumetric debit of bore hole on unit of power layer, led to the initial stratified pressure p_0.

Linearization method possesses one additional dignity, especially essential from logical point of view, it it is possible to include/connect into the common diagram of small parameter method. Specifically, thus understood this method L. S. Leybenzon [71].

A detailed account of small parameter method in the theory of nonstationary filtration and examples of its application/use can be found in the book p. Ya. Polubarinova-Kochina [94]; the sequential application/use of a small parameter method to the problems of investigating bore holes given in book [35]. a

2. The linearization and small parameter method are distinguished in terms of unwieldiness, especially in connection with the study of motion in the limited field. In order to go around this difficulty, just as in the problems of elastic mode/conditions in the filtration of gas, it is possible to search for approximate solution by the method of integral correlations. Let us examine for an example the problem of the exhaustion of gas deposit with a radius of R, by the operable single centrally arranged/located bore hole. Under usual assumptions the problem is reduced to the solution to equation

$$\frac{\partial p}{\partial t}=\frac{k}{2m\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p^2}{\partial r}\right) \qquad (V.3.4)$$

under the conditions of

$$p(r,\ 0)=p_0;\quad \frac{\partial p(R,\ t)}{\partial r}=0;\quad \frac{2\pi r_0 kp}{\mu}\frac{\partial p(r_0 t)}{\partial r}=q. \qquad (V.3.5)$$

From the last boundary condition (V.3.5) it follows that during approach/approximation to bore hole pressure change can be asymptotically presented expression [1]

$$p^2 \approx f(r,\ t) - \frac{q\mu}{\pi k} \ln r, \qquad \text{(V.3.6)}$$

where f (r, t) do not have a special feature/peculiarity with $r \to 0$.

FOOTNOTE [1]. This expression, obviously, it is possible to use if and only if its right side is positive (comp. the corresponding part in

chapter IV about self-similar problems). ENDFOOTNOTE.

Page 133.

In accordance with by this we will determine the pressure distribution in the form:

$$p^2 = \frac{q\mu}{\pi k}\left[\ln\frac{r}{l(t)} + a_0(t) + a_1(t)\frac{r}{l(t)} + \dots \right.$$
$$\left. \dots + a_n(t)\frac{r^n}{l^n(t)}\right] \quad (r \leqslant l(t));$$
$$p^2 = p^2(R) \qquad (r \geqslant l). \qquad \text{(V.3.7)}$$

Motion, as usual, is subdivided into two stage: on first stage

$l(t) < R$ (disturbance/perturbation still with P reached the boundaries of layer) and $p(l, t) = p_0$; at the second stage $l(t) = R$.

Let us multiply equation (V.3.4) by r^n and integrate from r_0 to $l(t)$.

With r = 1 after simple conversions, it has

$$\frac{d}{dt}\int_{r_0}^{l(t)} rp\,dr = \frac{k}{2m\mu}\left(r\frac{\partial p^2}{\partial r}\right)_{r_0}^{l(t)} - lp_0\frac{dl}{dt} = -\frac{q}{2\pi m} - lp_0\frac{dl}{dt} \quad (V.3.8)$$

(equation of material balance); with n > 1

$$\frac{d}{dt}\int_{r_0}^{l(t)} r^n p\,dr = \frac{k}{2m\mu}\int_{r_0}^{l(t)} r^{n-1}\left(\frac{\partial}{\partial r} r\frac{\partial p^2}{\partial r}\right)dr - l^n p_0\frac{dl}{dt} =$$

$$= \frac{k}{2m\mu}\left(r^n\frac{\partial p^2}{\partial r}\right)_{r_0}^{l} - \frac{k(n-1)}{2m\mu}\int_{r_0}^{l} r^{n-1}\frac{\partial p^2}{\partial r}\,dr - l^n p_0\frac{dl}{dt} =$$

$$= \frac{k}{2m\mu}\left(r^n\frac{\partial p^2}{\partial r}\right)_{r_0}^{l} - \left(\frac{k(n-1)^2}{2m\mu} r^{n-1}p^2\right)_{r_0}^{l} +$$

$$+ \frac{k(n-1)^2}{2m\mu}\int_{r_0}^{l} r^{n-2}p^2\,dr - l^n p_0\frac{dl}{dt}. \quad (V.3.9)$$

Usually these expressions it is possible substantially to simplify, taking into account that into being of interest cases r_0 / l << 1. therefore r_0 it is possible to everywhere rely equal to zero, which we have in form, that $\frac{k}{2\mu} r_0 \frac{\partial p^2(r_0, t)}{\partial r} = \frac{q}{2\pi}$, and the work of $r_0^n p(r_0, t)$ is small. Hence we will obtain instead of (V.3.9)

$$\frac{d}{dt}\int_{r_0}^{l(t)} r^n p\, dr = -\frac{k(n-1)^2}{2m\mu} l^{n-1} p_0^2 + \\ + \frac{k(n-1)^2}{2m\mu}\int_{r_0}^{l} r^{n-2} p^2\, dr - l^n p_0 \frac{dl}{dt}. \qquad \text{(V.3.10)}$$

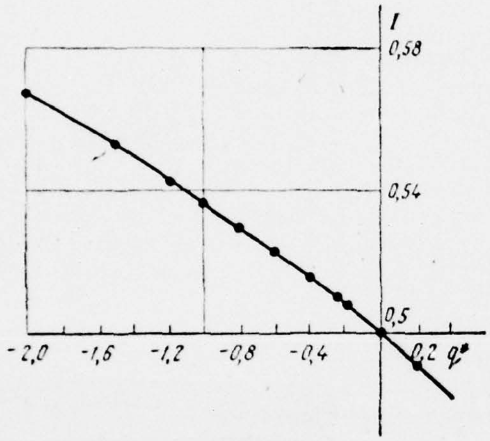
Let us find solution in the first approximation, by set/assuming $a_i(t) = 0$ $(i = 2, 3, \ldots, n)$. From conditions $p(l, t) = p_0$, $\partial p(l, t)/\partial r = 0$ let us determine for the first phase of the motion of $a_0(t) = p_0^2 \frac{\pi k}{\mu q} + 1$, $a_1(t) = -1$, and the unknown

solution takes the form:

$$p_0^2 - p^2 = -\frac{q\mu}{\pi k}\ln\frac{r}{l(r)} + \frac{q\mu(r-l)}{\pi k l(t)}. \qquad \text{(V.3.11)}$$

Fig. V.3.

$$\frac{d}{dt}\int_{r_0}^{l(t)} r\sqrt{p_0^2+\frac{\mu q}{\pi k}\left(\ln\frac{r}{l}+1-\frac{r}{l}\right)}\,dr=-\frac{q}{2\pi m}-lp_0\frac{dl}{dt}.$$



From (V.3.8) we have

Without making significant error, it is possible to replace in integral r_0 by 0 [1].

FOOTNOTE [1]. The appearing in this case small imaginary addition we reject/throw. ENDFOOTNOTE.

Then

$$\frac{d}{dt}\left[l^2 p_0 \int_0^1 u \sqrt{1+\frac{\mu q}{\pi k \rho_0^2}(\ln u + 1 - u)}\,du\right] = -\frac{q}{2\pi m} - p_0 l \frac{dl}{dt} \quad \text{(V.3.12)}$$

in the extent/elongation of the first phase of motion.

Thus, in the extent/elongation of the first a phase of motion we have for l (t):

$$l^2\left(\int_0^1 u\sqrt{1+q^*(\ln u+1-u)}-{}^1/_2\right)=-\frac{kp_0}{2m\mu}q^*;\quad q^*=\frac{\mu q}{\pi k p_0^2}.$$

the dependence of integral I in (V.3.12) of the dimensionless parameter q * is shown in Fig. V.3. Thus,

$$l=c\sqrt{\varkappa t};\ \varkappa=\frac{kp_0}{m\mu};\ c=\sqrt{q^*/(2I-1)},$$

where the constant s depends on the only dimensionless parameter q * (see Fig. V.3).

Page 135.

With $t > R^2/c^2\varkappa$ let us place l ≡ R, and thus

$$p^2(r,\ t) = p_R^2 + \frac{\mu q}{\pi k}\ln\frac{r}{R} - \frac{\mu q\,(r-R)}{\pi kR}. \qquad \text{(V.3.13)}$$

. Here p_R - pressure on the outline of layer. It is clear that this pressure must change in time because of the exhaustion of layer.

In order to find the law of the change in p_R, we will use again the equation of material balance. By set/assuming in it 1 = R, we will obtain

$$\frac{d}{dt} R^2 \int_0^1 u \sqrt{p_R^2 + \frac{\mu q}{2\pi k} (\ln u + 1 - u)}\, du = - \frac{q}{2\pi m}$$

or

$$\frac{p_R}{p_0} \int_0^1 u \sqrt{1 + \frac{p_0^2}{p_R^2} q^* (\ln u + 1 - u)}\, du -$$
$$- \int_0^1 u \sqrt{1 + q^* (\ln u + 1 - u)}\, du = - \frac{q (t - t_1)}{2\pi m p_0^2 R^2}. \qquad \text{(V.3.14)}$$

. Here t_1 is torque/moment of the end of the first phase of motion. By knowing the dependence of integral

$$I(q^*) = \int_0^1 u\sqrt{1+q^*(\ln u+1-u)}\,du \qquad (V.3.15)$$

of parameter q * (see Fig. V.3), it is possible, using an equation (V.3.14), to build the dependence of aaaaaa on t. For a practice, however, sufficient accuracy gives the rough approximation when integral (V.3.15) simply is set/assumed equal to 1/2. Physically this is equivalent to the equating of mean pressure in layer to pressure on outline. As it follows from Fig. V.3, for real values of the dimensionless debit of bore hole q * (order of hundreds of portion/fractions) this is admissible with error less than 2 o/o. in this case,

$$p_R^2 = p_0^2 - \frac{q}{\pi m R^2}(t-t_1) \qquad (V.3.16)$$

and

$$p^2(r,\ t) = p_0^2 - \frac{q}{\pi m R^2}(t-t_1) + \frac{\mu q}{\pi k}\left(\ln\frac{r}{R}+1-\frac{r}{R}\right). \qquad (V.3.17)$$

. If we do not approach the precise satisfaction of the condition of impenetrability with $r = R$, then it is possible to reject/throw the last terms in expressions for a pressure (i.e. to place a_1 (t) = 0. The obtained in this case expression answers solution by the method of the sequential change of steady states. It was for the first time obtained by B. B. Lapuk [67, 68] and widely it is used in practical calculations.

Page 136.

It is possible try to build, being held the usual technique of the application/use of a method of integral correlations (see the

foregoing paragraph), the subsequent approach/approximations. In this case, however, for non-self-simulating motions coefficients of aaaa and a radius of the field of influence l it is necessary to find from complex nonlinear system of equations. The attempt to get rid of difficulties by means of the replacement of root in element of integration (V.3.15) by the first two members of its expansion actually means passage to the linearized theory of the flow of gas. Numerous works are executed by this method of the Lan Chang-Sin [64, 65].

3. From the given example it is clear that the application/use of a method of integral relationships to the problems of the filtration of gas justified or when we allow passage to those which were linearized approach/approximation. In other words, more advantageous to achieve approach/approximation in more complex functions, but to be limited to a minimum number of free parameters.

One of the method of this approach/approximation entails the use of self-simulating solutions by analogy with the fact, as in boundary-layer theory are used the self-simulating solutions of Fckner-Skan (method of Kochin-Loytsyanskiy [60], another method is stated in following paragraph.

In chapter IV, were given (in connection with the equivalent problem of the filtration of ground water) the self-simulating solutions of the one-dimensional problems of the isothermal filtration of ideal gas, described by equation

$$\frac{\partial p}{\partial t}=\frac{a^2}{x^s}\frac{\partial}{\partial x}x^s\frac{\partial p^2}{\partial x} \qquad (V.3.18)$$

(s = 0; 1; 2 - respectively for flat/plane, axisymmetrical and centrally symmetric motions). These solutions were constructed for the case when on the boundary of layer (with x = 0) is assigned either pressure

$$p(0,\ t)=\Phi(t), \qquad (V.3.19)$$

that possibly only in the case of plane-parallel motion or the flow of gas

$$\lim_{x\to 0}\left(x^s\frac{\partial p^2}{\partial x}\right)=-\Psi(t). \qquad (V.3.20)$$

Motions are self-simulating during the specific combination of the initial and boundary conditions. Specifically, if occurs filling of the layer in which at first the pressure of gas by very small, so that it it is possible to consider equal to zero that the problem is self-simulating with arbitrary exponential functons f (t) or $\Psi(t)$:

$$\Phi(t) = \sigma t^{\alpha};\ \Psi(t) = \tau t^{\beta}, \qquad (V.3.21)$$

where σ, τ, α and β are some constants.

Page 137.

In the general case when the initial pressure in layer not is

equal to zero, motion is self-simulating only with $\alpha = 0$ and $\beta = 1/2$ (s are 1).

The skeletal diagram of the application/use of self-simulating solutions for the approximate solution of nonlinear problem entails the fact that is undertaken the one-parameter family of the self-simulating solutions, which correspond these initial and boundary conditions, and then this parameter is set/assumed equal to certain function of time, whereupon the form of this function is selected so that the differential equation of problem would be satisfied on the average. In other words, it is necessary that would be fulfilled certain integral correlation, which is the consequence of the initial problem.

It is obvious, there exists many methods of the introduction of the parameter to self-simulating solution and the variations of this parameter. Each of these methods leads to one or the other approximate solution of problem. Usually it cannot be previously said, which method of solution will turn out to be more successful.

Let us examine first the process of the flat/plane

one-dimensional filtration of gas into empty layer [equation (V.3.18) with s = 0 with zero initial condition]. Let on boundary of x = 0 law of a change in the pressure [equation (V.3.19)]. If f (t) = of σt^{α}, then solution self-simulating can be presented in the form:

$$p(x, t)=\sigma t^{\alpha} f(\xi, \lambda); \ \xi=\frac{x}{a}\sqrt{\frac{\alpha+1}{\sigma t^{\alpha+1}}}, \qquad \text{(V.3.22)}$$

where f (ξ, λ) - the solution to boundary-value problem

$$\frac{d^2 f^2}{d\xi^2}+\frac{1}{2}\xi\frac{df}{d\xi}-\lambda f=0; \quad \lambda=\frac{\alpha}{\alpha+1}; \quad f(0)=1; \ f(\infty)=0. \ \text{(V.3.23)}$$

Solution identical equal to zero outside final interval/gap $0 \leqslant \xi \leqslant \xi^*$ satisfies integral correlation

$$\int_0^{\xi^*} \xi f(\xi, \lambda)\, d\xi = 1/(1+\lambda).$$

. in Table 1 chapter IV (see also Fig. IV.4) were given the values of functions f (ξ, λ) for values λ, equal to 0.00; 0.05; ...; 1.00 and value of argument ξ, equal to 0.1 ξ *; 0.2 ξ * etc.

in the general case of arbitrary function f (t) the corresponding integral correlation assumes the form:

$$\frac{d}{dt} \int_0^{x^*(t)} xp(x, t)\, dx = a^2\Phi^2(t). \qquad \text{(V.3.24)}$$

. Here * (t) is a coordinate of the leading edge of the forward

movement of gas.

We will search for the approximate solution of the formulated problem in the form:

$$p(t)=\Phi(t)f\left[\frac{x}{a}\sqrt{\frac{\alpha(t)+1}{t\Phi(t)}},\ \lambda(t)\right];\ \lambda(t)=\frac{\alpha(t)}{\alpha(t)+1}.\ (V.3.25)$$

Page 138.

With f (t) = σt^{α}, α = const expression (V.3.25) passes to precise self-simulating solution. Therefore it is logical to expect that during functions (t), close to exponential, the selected idea will ensure good approach/approximation. After accepted expression (V.3.25), the solution of problem is reduced to the determination of only function λ (t). Determining it from integral correlation (V.3.24), we find

$$\lambda(t)=\frac{\left[t\Phi^2(t)-\int\limits_0^t\Phi^2(t)\,dt\right]}{\left[t\Phi^2(t)+\int\limits_0^t\Phi^2(t)\,dt\right]}. \qquad \text{(V.3.26)}$$

. After this through formula

$$\alpha(t)=\frac{\lambda(t)}{[1-\lambda(t)]}, \qquad \text{(V.3.27)}$$

we find α (t) and, using formula (V.3.25) and Table IV.1, we find the unknown solution. The coordinate of the leading edge of the advance of gas is determined in this case by correlation

$$x^*(t)=a\xi^*[\lambda(t)]\sqrt{\Phi(t)\,t\,[1-\lambda(t)]}. \qquad \text{(V.3.28)}$$

Let us examine an example of the solution of problem in the procedure presented.

Let us place f (t) = of $\Phi(t) = \sigma_m t^m + \sigma_n t^n, m < n.$ Then from formulas (V.3.26) and (V.3.27) we find

$$\alpha(t) = \frac{1}{2}\left(\frac{\sigma_m^2 t^{2m+1} + 2\sigma_m\sigma_n t^{m+n+1} + \sigma_n^2 t^{2n+1}}{\frac{1}{2m+1}\sigma_m^2 t^{2m+1} + \frac{2}{m+n+1}\sigma_m\sigma_n t^{m+n+1} + \frac{1}{2n+1}\sigma_n^2 t^{2n+1}} - 1\right).$$

From this expression it is directly evident that with small t $\alpha(t) \approx m$, and with large t $\alpha(t) \approx n$. This means that with motion

occurs the passage from one self-simulating motion to another.

During the calculation of a concrete/specific/actual example, let us place a = 1, m = 0, n = 1, $\sigma_m = \sigma_n = 1$. Then

$$\alpha(t) = \frac{1}{2}\left(\frac{1+2t+t^2}{1+t+{}^1/_3 t^2} - 1\right);$$
$$p(x,\ t) = (1+t)\, f\left[\frac{x\sqrt{1+\alpha(t)}}{\sqrt{t+t^2}},\ \lambda(t)\right].$$

Using table of functions f (ξ, λ), it is possible to calculate the distribution of pressure p (x, t) at different points in time. The results of calculations are shown in Fig. V.4.

Let us examine now the problem of filling of layer on the assumption that is assigned the flow of the gas through the bore hole

of the negligible radius: $\lim_{r\to 0}\left(r\frac{\partial p^2}{\partial r}\right) = -\Psi(t)$.

end section.

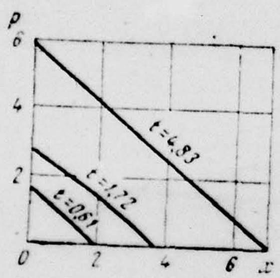


Fig. V.4.

If :ps (t) = $\psi(t) = \tau t^{\beta}$, then the sclution of problem self-similarly can be presented in the form (see Chapter IV, § 3):

$$p(x,\ t) = \tau^{1/2} t^{\beta/2} f_1(\xi,\ \lambda); \quad \lambda = \beta/(\beta+2);$$
$$\xi = r\,[4a^2 \tau t^{\beta+2}]^{-1/4} (\beta+2)^{1/2}; \qquad \text{(V.3.29)}$$
$$f_1(\infty) = 0; \quad \lim_{\xi \to 0} \xi \frac{df_1^2}{d\xi} = 1.$$

Function f_1 $(\xi,\ \lambda)$ becomes zero with $\xi \geqslant \xi_1(\lambda)$ and satisfies integral relationship/ratio (IV.2.32):

$$\int_0^{\xi_1(\lambda)} \xi f_1(\xi,\ \lambda)\, d\xi = \frac{1}{1+\lambda}. \qquad \text{(V.3.30)}$$

Let us find the approximate solution of the formulated problem in the form:

$$p = [\Psi(t)]^{1/2} f_1 \left[\frac{r}{a} \left(\int_0^t \sqrt{\Psi(t)}\, dt \right)^{-1/2},\ \lambda(t) \right]. \tag{V.3.31}$$

As it is not difficult to see, this approximate representation of solution satisfies the initial condition and condition at infinity with any λ (t), function λ (t) we count to be determined. For its determination we will use integral relationship/ratio

$$\frac{d}{dt}\int_0^{r^*(t)} rp(r,\ t)\,dr = a^2\Psi(t), \qquad \text{(V.3.32)}$$

where r * (t) - the boundary of the region of gas permeation; $p(r,t)\geqslant 0$ with $r\leqslant r^*(t)$; $p(r,t)\equiv 0$ at $r\geqslant r^*(t)$. After some calculations we will obtain

$$\lambda(t)+1=\sqrt{\Psi(t)}\int_0^t\sqrt{\Psi(t)}\,dt\left[\int_0^t\Psi(t)\,dt\right]^{-1}. \qquad \text{(V.3.33)}$$

Formulas (V.3.31) and (V.3.33) make it possible to express approximate solution by the tabulated function f_1 (ξ, λ). When

problem admits a precise self-similar solution, approximate solution coincides with precise.

There is two reasons for which we were restricted to the use of self-similar solutions only for the approximate solution of problems with the zero initial pressure of gas. in-vpervyx, only in this case solution is self-similar with any (constants) α and β, i.e., there is a one-parameter family of solutions. First, precisely the problems of filtration with the zero initial pressure of gas represent the special complexity for the methods of and low parameter, since the expansion in terms of the parameter of $\nu = 1 - (p_1/p_0)^2$ becomes in this case illegal.

Page 140.

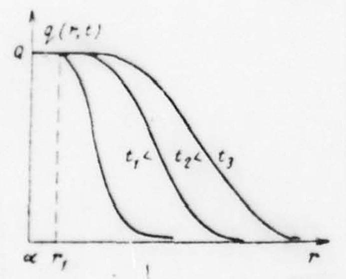


Fig. V.5.

4. Modification of the method of integral relationship/ratios for the case of infinite layer.

During the solution of the nonlinear problems of different form, the consecutive application/use of a method of integral relationship/ratios in standard diagram leads to the cumbersome calculations, the facts are more complex, the larger the number of unknown coefficients in approximation. Considerable simplification it is possible to achieve by this selection of the approaching functions in order that the necessary accuracy would be obtained already in one of the first approximations (but virtually - in the first approximation,). In other words, problem lies in the fact that, in order to almost "guess" solution. Great aid in this case it can show preliminary qualitative investigation. This investigation already was described into 4 chapters IV for a self-similar problem of the inflow of gas to the hole, put into service with constant flow rate. In this case, it turned out that the motion near hole rapidly is stabilized, so that the mass flow rate of the gas through the coaxial with hole cylindrical surface virtually is not changed up to certain distance from hole (Fig. V.5). There, where the flow rate begins to be changed considerably, changes in the pressure of gas with respect to initial value are small, and equation of motion it is possible to linearize. This makes it possible to actually consider that the gas

flow depends on distance and time just as in the appropriate linearized problem, whereupon nearness true and the "linearized" distributions of flow rate is caused by the smallness of usually being encountered values of flow rate (by smallness of values λ in the designations of chapter IV). By the method of integral relationship/ratios taking into account the indicated considerations it is possible to obtain the simple and sufficiently exact solutions of series of problems.

1. Let us examine the axisymmetric motion of compressible liquid in the deformed-medium, which follows to the law of darcys. For our purposes will conveniently introduce in clear form the value of the mass fluid flow rate at this torque/moment through surface r = const of the single height/altitude:

$$q = r\rho u = -r\frac{k}{\mu}\rho\frac{\partial p}{\partial r}. \qquad \text{(V.4.1)}$$

Equation (V.4.1) together with the equation of continuity

$$\frac{\partial (m\rho)}{\partial t}+\frac{1}{r}\frac{\partial q}{\partial r}=0 \qquad (V.4.2)$$

and by equations

$$\rho=\rho(p),\quad k=k(p),\quad \mu=\mu(p),\quad m=m(p) \qquad (V.4.3)$$

composes the locked system of equations of motion.

Page 141.

By introducing function

$$P(p)=\int_0^p \frac{k\rho}{\mu}\,dp, \qquad (V.4.4)$$

being the analog of the function of Leybenzon in the theory of the

filtration of gas, it is possible to lead system (V.4.1), (V.4.2.) to the standard form:

$$\frac{\partial P}{\partial t}+\frac{\varkappa(P)}{r}\frac{\partial q}{\partial r}=0;\quad q=-r\frac{\partial P}{\partial r},\qquad (\text{V}.4.5)$$

where

$$\varkappa(P)=\left(\frac{d(m\rho)}{dP}\right)^{-1}=\frac{k\rho}{\mu}\left[\frac{d(m\rho)}{dp}\right]^{-1}\qquad (\text{V}.4.6)$$

is a variable coefficient of piezoconductivity.

System (V.4.5) can be written in the form one equation for P. In this form it is possible to write the system of equations of motion during the elastic mode/conditions of the equation of the isothermal filtration of gas and equation of filtraticn in the nonlinear deformed medium, so that system (V.4.5) describes sufficiently common/general/total situation.

We will assume that is examined certain disturbance/perturbation of steady state, which appears on the internal boundary of system (in hole). Most frequently this disturbance/perturbation consists of the fact that is assigned the determined law of a change in the selection from hole. Therefore we will consider that the problem for equations (V.4.1) takes the form:

$$q(r,\ 0)=q_0=\mathrm{const};\quad q(a,\ t)=Q(t)$$
$$(a\leqslant r<\infty,\ 0<t<\infty). \qquad (V.4.7)$$

. Conditions (V.4.7) correspond to the most important for application/appendices problem of unsteady inflow to hole (a is a radius of well).

Simplest is the case, when the flow rate in the hole of changes abruptly. In this case, from hole, begins to be propagated the wave of a change in the flow rate, and distribution q (r, t) assumes the form, shown for the consecutive torque/moments of time in Fig. V.5. It is characteristic in this case that the flow rate retains constant values near hole and at removal/distance from it only into certain intermediate region occurs it abrupt change.

This character of change q (r, t) occurs with all being of

interest forms of the dependence of $\varkappa(P)$. In the simplest case with of $\varkappa(P) = \varkappa = \text{const}$, Q = const, we have

$$q(r,\ t) = Q\exp\left(-\frac{r^2}{4\varkappa t}\right) \quad (4\varkappa t \gg a^2). \qquad \text{(V.4.8)}$$

Page 142.

Is logical therefore to attempt to find the approximate solution of problem (V.4.1) - (V.4.2), set/assuming

$$q(r,\ t) = q_0 + (Q - q_0)\exp\left(-\frac{r^2}{l^2}\right), \qquad \text{(V.4.9)}$$

where $l = l(t)$ - the parameter, selected so that in the very best manner to satisfy certain supplementary condition, which will be

given below. If it is desirable to consider also the finiteness of a radius of hole and change in time of the output of hole Q (t), then it is convenient to accept

$$q(r, t) = q_0 + (Q - q_0) \exp \frac{a^2 - r^2}{l^2}. \qquad \text{(V.4.10)}$$

Unlike (V.4.9) the expression (V.4.10) is not precise even for the case of elastic mode/conditions. It, however, is convenient in that relation, which makes it possible to considerably simplify calculation, providing sufficiently good approach/approximation.

For determining function l (t) is used integral relationship/ratio

$$\frac{1}{2}\frac{d}{dt}\int\limits_a^{\infty}(r^2-a^2)\left[\frac{q(r,\ t)-q(r,\ 0)}{r}\right]dr=\int\limits_a^{\infty}\varkappa(\dot{P})\frac{\partial q}{\partial r}dr. \qquad \text{(V.4.11)}$$

Expression (V.4.10) is convenient in that relation, that after its substitution the equation (V.4.11) accepts sufficiently simple form.

Before passing to the consideration of examples, let us make one generality. Theselection of flow rate q (r, t) as the function for which is assigned the distribution of relatively simple form, is not accidental. It is possible to show that in the problems in which on the boundaries of the region of motion are fixed values q, the distribution of the flow rate comparatively barely it depends on the form of equations of motion, remaining qualitatively the same as for the problems of elastic mode/conditions. For this reason the distribution of flow rate of q (r, t) is sufficiently easy "to guess"

with the required accuracy. Here we will use this possibility only for motion in uniform infinite layer; however, the same approach let us use to heterogeneous layers and the layers of the final duration.

2. As the first example let us take the problem from the region of the theory of elastic mode/conditions, examined still Masket [78], of inflow to the hole of terminal radius, released with constant output. In this case of $P = \rho k p/\mu$; $\varkappa = \varkappa_0 = kK/m\mu = \text{const}$; $Q = \text{const}$; q $(r, 0) = 0$. Substituting for q expression (V.4.10), from (V.4.11) we obtain

$$\frac{d}{dt}\left[\frac{l^2}{4}+\frac{a^2}{4}e^{a^2/l^2}\,\mathrm{Ei}\left(-\frac{a^2}{l^2}\right)\right]=\varkappa_0 \qquad \text{(V.4.12)}$$

or

$$t=\frac{l^2}{4\varkappa_0}\left[1+\frac{a^2}{l^2}e^{a^2/l^2}\,\mathrm{Ei}\left(-\frac{a^2}{l^2}\right)\right]. \qquad \text{(V.4.13)}$$

Page 143.

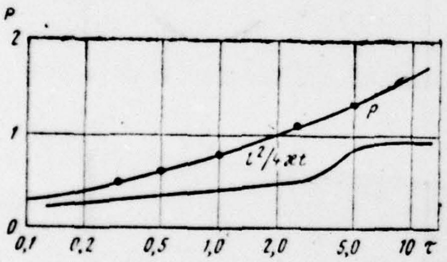


Fig. V.6.

Integration of the first equation of system (V.4.1) taking into account condition p (∞,t) = p (r, 0) = 0 gives

$$p(r,\ t)=\frac{\mu Q}{k\rho}\,\mathrm{Ei}\left(-\frac{r^2}{l^2}\right). \qquad \text{(V.4.14)}$$

Expressions (V.4.14) and (V 4.13) represent in parametric form unknown dependence p (r, t). Figure V.6 shows the dependence of the relation of $l^2/(4\varkappa t)$ and dimensionless depression in the hole of $P=p(a,\ t)\frac{k\rho}{\mu Q}$, on dimensionless time of $\varkappa t/a^2=\tau$; the obtained solution will agree well with the exact solution of Masket [78].

Let us examine now several problems of the filtration of gas.

Let us assume that in initial state of motion no completely, q (r, 0) = 0, and initial value of the function of Leybenzon P (r, 0) = P_0 it is equal in all points, P (r, 0) = P_0 = const. In this case,

utilizing the first equation of system (V.4.5) and relationship (V.4.11) and being limited to first-order correction for a change of the $\varkappa(P)$

$$\varkappa(P)=\varkappa(P_0)\,[1+\eta_0\,(P-P_0)], \tag{V.4.15}$$

easy to present integral relationship/ratio (V.4.11) in the form:

$$\frac{d}{dt}\left[\frac{Ql^2}{4}+\frac{Qa^2}{4}e^{a^2/l^2}\,\mathrm{Ei}\left(-\frac{a^2}{l^2}\right)\right]=$$
$$=\varkappa_0\left\{Q+\frac{1}{2}\eta_0Q^2\left[e^{a^2/l^2}\,\mathrm{Ei}\left(-\frac{a^2}{l^2}\right)-e^{2a^2/l^2}\,\mathrm{Ei}\left(-\frac{2a^2}{l^2}\right)\right]\right\}. \tag{V.4.16}$$

In turn, for P (r, t) from (V.4.1) and (V.4.8) is obtained the expression

$$P(r,\ t) = P_0 + \frac{1}{2} Q\, e^{a^2/l^2}\, \mathrm{Ei}\left(-\frac{r^2}{l^2}\right). \qquad (V.4.17)$$

In the in practice interesting cases the expression (V.4.16) accomplishes without the special work tc simplify.

If we consider only sufficiently long times $l^2/a^2 >> 1$, then equation (V.4.16) will seem in the form:

$$\frac{1}{4}\frac{d}{dt}\left(Ql^2 - Qa^2\ln\frac{l^2}{a^2}\right) = \varkappa_0 Q\left(1 - \frac{1}{2}\eta_0 Q\ln 2\right). \quad \text{(V.4.18)}$$

. Page 144.

During the conclusion/derivation of this expression, it is registration/accounting alsc that virtually in all cases of $\eta_0 Q \ll 1$. Frcm (V.4.18) follows the expression for the time:

$$t = \frac{l^2 - a^2 \ln (l^2/a^2)}{4\varkappa_0 [1 - {}^1/_2\, \eta_0 Q \ln 2]}, \qquad \text{(V.4.19)}$$

while from (V.4.17) is a formula

$$P(r,\ t) = P_0 + \frac{1}{2} Q\, e^{a^2/l^2}\, \mathrm{Ei}\left(-\frac{r^2}{l^2}\right). \qquad \text{(V.4.20)}$$

With long times it is possible, by disregarding the terms of order $(a^2/l^2)\ \ln\ (l^2/a^2)$, to present (V.4.20) in the form:

$$P(r,\ t) = P_0 + \frac{1}{2} Q\, \mathrm{Ei}\left(-\frac{r^2}{4\varkappa_0 t\, [1 - {}^1/_2\, \eta_0 Q \ln 2]}\right). \qquad \text{(V.4.21)}$$

This formula very close to the analogous formula of the theory of elastic mode/conditions and coincides with it, if we disregard the value of the product of $^{1}/_{2}\eta_0 Q \ln 2$ in comparison with unit. Therefore, by observing a change in the function of Leybenzon during the launching/starting of gas well with constant output, it is possible to define the parameters of layer in the same way as during elastic mode/conditions are determined the parameters of layer from pressure changes. The same result is obtained if we reference system (V.4.5) linearize in L. S. Leybenzon's method, i.e., after replacing variable coefficient of piezoconductivity $\varkappa(P)$ with a constant value of $\varkappa_0 = \varkappa(P_0)$. Component $^{1}/_{2}\eta_0 Q \ln 2$ is correction to the linearized theory.

3. For determining the parameters of layer from the tests of gas well for unsteady inflow in accordance with formula (V.4.21) it is necessary that the output of hole in the extent/elongation of tests would remain constant. Since the pressure on the face of hole in this case greatly changes, for maintaining constant output it is necessary to use special measures, are always attained. Considerably simpler to conduct experiment, leaving the hydraulic resistance of hole by constant. In this case the output of hole proves to be the function

cf driving in pressure

$$Q = Q(p_a). \tag{V.4.22}$$

This function (the discharge characteristic of hole) can be determined independently, and it it is possible to consider known. Moreover, in one case it can be easily calculated; if in immediate proximity to the face of hole in it is established/installed diaphragm sufficient small passage cross section, then the outflow of gas into hole bears critical character; in this case the flow rate is approximately proportional to pressure on face

$$Q = cp_a. \tag{V.4.23}$$

. it is possible to obtain the approximate solution, which makes it possible to determine the characteristics of layer according to observations of pressure change if output also changes in accordance with relationship/ratios (V.4.23) or (V.4.22).

Page 145.

Let us examine sufficiently great significance of the time: $l^2 \gg a^2$. Then equation (V 4.18) will take the form:

$$\frac{d}{dt}\frac{Q(t)\,l^2}{4} = \varkappa_0 Q(t)\left[1 - \frac{1}{2}\eta_0 Q(t)\ln 2\right]. \qquad (V.4.24)$$

utilizing (V.4.22), it is possible to connect the value of the function of Leybenzon in hole P (a,t) with output Q. Then from (V.4.17) follows equation

$$P(a,\ t) = P_0 + \frac{1}{2}Q\,[P(a,t)]\,\mathrm{Ei}\left(-\frac{a^2}{l^2}\right). \qquad (V.4.25)$$

System of equations (V.4.25) and (V.4.24) is easy to solve approximately, assuming that a change in output Q (t) occurs sufficiently slowly. Let us note first of all, that Q (t) monotonically decreases from value Q_0 = Q (0), the corresponding torque/moment launching/starting

$$Q_0 = Q(p_0). \qquad \text{(V.4.26)}$$

By taking into account this fact and by utilizing an equation (V.4.24), it is possible easily to obtain the estimation:

$$\left[1-\frac{1}{2}\eta_0 Q(t)\ln 2\right]\frac{Q_0}{Q(t)} \geqslant \frac{l^2}{4\varkappa_0 t} \geqslant 1-\frac{1}{2}\eta_0 Q_0 \ln 2. \qquad (V.4.27)$$

From (V.4.27) it follows that with a small relative error in determination l^2/a^2 - order $\ln [Q_0/Q(t)]/\ln (4\varkappa_0 t/a^2)$ - it is possible to rely

$$\frac{l^2}{a^2} = \frac{4\varkappa_0 t}{a^2}. \qquad (V.4.28)$$

If we substitute expression (V.4.28) in (V.4.25), then will be obtained the equation, which defines P (a, t) as implicit function of

time. This equation it is convenient to convert to the forms:

$$\chi(p_a) = \frac{Q_0}{2P_0} \ln \frac{2{,}25\varkappa_0 t}{a^2}, \qquad (V.4.29)$$

where

$$\chi(p_a) = \frac{P_0 - P(a,\ t)}{P_0} \frac{Q_0}{Q\,[P(a,\ t)]} \qquad (V.4.30)$$

is a dimensionless function of driving in pressure, determined by the dependence of the function of Leybenzon and output of hole of pressure; the dependence of $\chi(p_a)$ can be previously determined for this hole. Formula (V.4.29) shows that in experiments with critical

outflow, the value of χ so depends on time as dimensionless pressure under conditions of elastic mode/conditions. Therefore, by representing the experimental data in the coordinates of $\chi(p_a)$ — ln t, it is possible to determine the parameters of layer according to usual procedure (see Chapter III, 4).

With wish the obtained expressions can be refined, by introducing corrections in equations (V.4.28) for η^2. In practice, however, with the usually being encountered values of output in this correction there is no need.

end section.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER FTD-ID(RS)T-1860-76	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE THEORY OF THE UNSTEADY FILTRATION OF LIQUID AND GAS		5. TYPE OF REPORT & PERIOD COVERED Translation
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) G. I. Barenblatt, V. M. Yentov, V. M. Ryzhik		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Foreign Technology Division Air Force Systems Command U. S. Air Force		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 1972
		13. NUMBER OF PAGES 931
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

13

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)